

CHOW QUOTIENTS OF GRASSMANNIANS II

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§1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. Let $\mathbb{P}(r, n)$ be the space of ordered n -tuples of linear hyperplanes in \mathbb{P}^{r-1} . Let $\mathbb{P}^\circ(r, n) \subset \mathbb{P}(r, n)$ be the open subset which are in linear general position. PGL_r acts freely on $\mathbb{P}^\circ(r, n)$, let $X(r, n)$ be the quotient. $X(2, n)$ is usually denoted $M_{0,n}$. It has a compactification $M_{0,n} \subset \overline{M}_{0,n}$, due to Grothendieck and Knudsen, with many remarkable properties:

1.2. PROPERTIES OF $M_{0,n} \subset \overline{M}_{0,n}$.

- (1.2.1) $\overline{M}_{0,n}$ has a natural moduli interpretation, namely it is the moduli space of stable n -pointed rational curves.
- (1.2.2) Given power series $f_1(z), \dots, f_n(z)$ which we think of as a one parameter family in $M_{0,n}$ one can ask: What is the limiting stable n -pointed rational curve in $\overline{M}_{0,n}$ as $z \rightarrow 0$? There is a beautiful answer, due to Kapranov [Ka1], in terms of the Tits tree for PGL_2 .
- (1.2.3) $M_{0,n} \subset \overline{M}_{0,n}$ has a natural Mori theoretic meaning, namely it is the log canonical model, [KM]. In particular the pair $(\overline{M}_{0,n}, \partial\overline{M}_{0,n})$ has log canonical singularities (a natural generalisation of toroidal).

In fact in (1.2.3) the pair has normal crossing, but we write the weaker form as this is what there is a chance to generalize. It is natural to wonder

1.3. QUESTION. Is there a compactification $X(r, n) \subset \overline{X}(r, n)$ which satisfied any or all of the properties of (1.2)?

1.4. CHOW QUOTIENTS OF GRASSMANNIANS. There is an identification

$$X(r, n) = G^0(r, n)/H,$$

the so called Gelfand–Macpherson correspondence (3.3), where $G(r, n)$ is the Grassmannian of r -planes in \mathbb{A}^n , $H = \mathbb{G}_m^n$ is the standard diagonal torus and $G^0(r, n) \subset G(r, n)$ is the open subset of r -planes which project isomorphically onto \mathbb{A}^I for any subset $I \subset N$ with cardinality $|I| = r$, where $N = \{1, \dots, n\}$. In [Ka], Kapranov has introduced a natural compactification, the so called Chow quotient

$$X(r, n) = G^0(r, n)/H \subset G(r, n)//H := \overline{X}(r, n).$$

See (2.2) for a review of Chow quotients. Here we note only that Kapranov defined a natural flat family

$$(1.4.1) \quad p : (\mathcal{S}, \mathcal{B}) \rightarrow \overline{X}(r, n)$$

of pairs of schemes with boundary, his so-called family of visible contours, generalising the universal family over $\overline{M}_{0,n}$, and Lafforgue in [La] gave a precise description of the fibres (S, B) , showing in particular each pair has

toroidal singularities (throughout the paper B will always indicate a boundary, i.e. a Weil divisor, on a space clear from context).

$\overline{X}(r, n)$ satisfies the first two properties of (1.2), but the third fails except possibly in the exceptional cases $(2, n)$, $(3, 6)$, $(3, 7)$, $(3, 8)$ (and those obtained from these by a canonical duality). We conjecture that in these cases the Chow quotient is indeed the log canonical model, and speculate a relationship to the exceptional root systems, see (1.14). Let $\tilde{X}(r, n) \rightarrow \overline{X}(r, n)$ be the normalization. In (2.11), we follow Lafforgue and introduce the third modification $\overline{X}_L(r, n)$ that sits between $\tilde{X}(r, n)$ and $\overline{X}(r, n)$ (we distinguish between these spaces in the interests of precision, in fact the minor differences will be for our purposes unimportant).

1.5. MODULI INTERPRETATION AS IN (1.2.1). $\overline{X}(r, n)$ is a natural moduli space of semi log canonical pairs (the natural higher dimensional Mori theoretic generalisation of stable pointed curves). This is a recent result of Hacking, [Ha]. We observed the same result independently. Our proof, which is based on [La], will appear elsewhere.

1.6. REALIZATION VIA TITS BUILDINGS AS IN (1.2.2). Let $R = k[[z]]$ and K its quotient field. Throughout the paper k is a fixed algebraically closed field. $V = k^r$ and $V_T = V \otimes_k T$ for a k -algebra T . We write \mathcal{P} for projective spaces of quotients (or equivalently, hyperplanes), \mathbb{P} for spaces of lines. Begin with a collection

$$(1.6.1) \quad \mathcal{F} := \{f_1, \dots, f_n\} \subset V_K$$

any r of which are linearly independent, and thus give a K -point of $X(r, n)$. We think of \mathcal{F} as the equations of n 1-parameter families in \mathbb{P}^{r-1} . Following [Ka1] (where the case $r = 2$ is treated) we ask:

1.7. QUESTION. *What is the limit as $z \rightarrow 0$, i.e. in the pullback of the family (1.4.1) along the associated R -point of $\overline{X}(r, n)$, what is the special fibre?*

We give a canonical solution, in terms of the Tits building \mathcal{B} . Here is a quick version, further details are given at the end of this introduction, see (1.21). Proofs and further related results are given in §4–§6. Recall \mathcal{B} is the set of equivalence classes of R -lattices in V_K (i.e. free R -submodules of rank r) where M and N are equivalent if there exists $c \in K^*$ such that $M = cN$. A subset $Y \subset \mathcal{B}$ is called *convex* if it is closed under R -module sums, i.e. $[M_1], [M_2] \in Y$ implies $[M_1 + M_2] \in Y$. For $Y \subset \mathcal{B}$ we write $[Y]$ for its convex hull, which is finite if Y is, see (4.11).

For a lattice M , and a non-trivial subset $\Theta \subset V_K$ (e.g. an R -submodule or an element) we can find unique $a > 0$ so that $z^a \Theta \subset M$, $z^a \Theta \not\subset zM$. We define $\Theta^M := z^a \Theta \subset M$. Let $\Theta^{\overline{M}} \subset \overline{M} := M/zM$ denote the image of the composition

$$\Theta^M \subset M \twoheadrightarrow M/zM = \overline{M}.$$

We call $[M]$ *stable* if $\mathcal{F}^{\overline{M}}$ contains $r + 1$ elements in linear general position. Let $\text{Stab} \subset [\mathcal{F}]$ be the set of stable classes. Stab is finite, see (5.22).

1.8. DEFINITION. For a finite $Y \subset \mathcal{B}$, let \mathbb{S}_Y be the *join* of projective bundles $\mathcal{P}(M)$, $[M] \in Y$ – i.e. fix one and take the closure of the graph of the product of the birational maps from this projective bundle to all the others.

Let $Y \subset \mathcal{B}$ be any finite convex subset containing Stab (for example take the convex hull $Y = [\text{Stab}]$). Let $\mathbb{B}_i \subset \mathbb{S}_Y$ be the closure of the hyperplane

$$\{f_i = 0\} \subset \mathcal{P}(V_K) \subset \mathbb{S}_Y$$

on the generic fibre of $p : \mathbb{S}_Y \rightarrow \text{Spec}(R)$. Let $\mathbb{B} = \sum \mathbb{B}_i$, and $S_Y \subset \mathbb{S}_Y$ the special fibre. Let V_n be the standard k -representation of S_n (i.e. elements in k^n whose coordinates sum to zero).

1.9. THEOREM. $\mathbb{S}_Y, \mathbb{B}_i$ are non-singular and the divisor $S_Y + \mathbb{B}$ has normal crossings. The 1-forms $d\log(f_i/f_j)$ define globally generating sections of the vector bundle $\Omega_{\mathbb{S}/R}^1(\log \mathbb{B})$. The image of the associated map

$$\mathbb{S}_Y \rightarrow \text{Spec}(R) \times G(r-1, n-1)$$

is $\mathcal{S} \rightarrow \text{Spec}(R)$, the pullback of the family (1.4.1) along the R -point of $\overline{X}(r, n)$ defined by \mathcal{F} . In particular the relative log canonical bundle $K_{\mathbb{S}_Y} + \mathbb{B}$ is relatively globally generated and big, and $\mathbb{S}_Y \rightarrow \mathcal{S}$ is the relative minimal model, and crepant.

1.10. HISTORICAL REMARK: By [Mu, 2.2], if Y is convex then $p : \mathbb{S}_Y \rightarrow \text{Spec}(R)$ is semi-stable, i.e. \mathbb{S}_Y is non-singular, and the closed fibre S_Y has smooth irreducible components and normal crossings. Mustafin remarks after the proof that the join probably represents some natural functor but he prefers the explicit join construction. This functor (see (5.3)) is introduced in Definition 4 of [Fa], and Faltings proves it represents the join. Faltings attributes the functorial description to Deligne, to whom Mustafin also refers. We will refer to \mathbb{S}_Y as the Deligne scheme, or Deligne functor.

For the definition and basic properties of bundles of relative log differentials see §9. We note that the crepant semi-stable model $(\mathbb{S}_Y, \mathbb{B})$ is in many ways preferable to its minimal model $\mathcal{S} \rightarrow \text{Spec}(R)$. For example dropping the last hyperplane induces a natural regular birational map

$$\mathbb{S}_{[\text{Stab}(\mathcal{F})]} \rightarrow \mathbb{S}_{[\text{Stab}(\mathcal{F}')]}$$

for $\mathcal{F}' = \mathcal{F} \setminus \{f_n\}$, but for $r \geq 3$ the associated rational map between minimal models is not in general regular. There are examples where regularity fails already with $(r, n) = (3, 5)$.

The special fibres (S_Y, B) and (S, B) can be canonically described in terms of the membrane $[\mathcal{F}] \subset \mathcal{B}$. We turn to this in (1.21) below, but wish first to discuss (1.2.3):

1.11. LOG CANONICAL MODEL AS IN (1.2.3). Let M be a smooth variety over the complex numbers, and let $M \subset \tilde{M}$ be a compactification, with normal crossing divisorial boundary, B . The vector spaces

$$H^0(\tilde{M}, \omega_{\tilde{M}}(B)^{\otimes m})$$

turn out to depend only on M , and so give a canonical rational map, the so called m -plurilogcanonical map, to projective space. The finite generation conjecture of Mori theory implies that if for some m the map is an immersion, then for sufficiently large m , the closure of the image gives a compactification $M \subset \overline{M}$ independent of m , and with boundary, B having nice singularities, namely $K_{\overline{M}} + B$ is log canonical. We do not recall here the definition of log

canonical (see e.g. [KMM]) but note that one can think of it as a weakening of toroidal, the pair of a toric variety with its boundary being one example.

The initial motivation for this paper was the elementary observation, (2.20), that $X(r, n)$ is minimal of log general type (its first log canonical map is a regular immersion) and thus there is (conjecturally) a natural Mori theoretic compactification, the log canonical model. It is natural to wonder:

1.12. QUESTION. *What is the log canonical model $X(r, n) \subset \overline{X}_{lc}(r, n)$?*

We believe this compactification is of compelling interest, as it gives a birational model with reasonable boundary singularities of a compactification of $X(r, n)$ whose boundary components meet in absolutely arbitrary ways. In particular, were $\overline{X}(3, n)$ the log canonical model, it would give something like a canonical resolution of all singularities, see (1.19). Unfortunately, the two compactifications do not in general agree:

1.13. THEOREM. *$\tilde{X}(3, n)$ with its boundary fails to be log canonical for $n \geq 9$ (for $n \geq 7$ in characteristic 2). $\tilde{X}(4, n)$ is not log canonical for $n \geq 8$.*

We prove (1.13) in §3. Moreover, (3.16) shows that in general the pair $(\overline{X}(3, n), B)$ has arbitrary singularities. We note (1.13) is at variance with the hope expressed in [Ha], and one that we ourselves for a long time harbored, that the pair has toroidal singularities. Faltings and Lafforgue, [Fa], [La1] expressed the same hope for their compactification of $\mathrm{PGL}_r^n / \mathrm{PGL}_r$ (which is itself a Chow quotient in a natural way), but Lafforgue has shown this hope was false as well, [La, 3.28].

In the positive direction we speculate the two agree in the cases that remain:

1.14. CONJECTURE. *$X(r, n) \subset \overline{X}(r, n)$ is the log canonical model precisely in the cases*

$$(2, n), (3, 6), (3, 7), (3, 8)$$

and those obtained from these by the canonical duality [Ka, 2.3]

$$\overline{X}(r, n) = \overline{X}(n - r, n).$$

Moreover in these cases the pair $(\overline{X}(r, n), B)$ has toroidal singularities.

The numbers in (1.14) are of course very suggestive and it is natural to wonder if there is a connection with the exceptional root systems

$$D_n, E_6, E_7, E_8.$$

1.15. EQUATIONS AND SYZYGIES OF $\overline{M}_{0, n}$. The closure of $X(2, n)$ in the first log canonical immersion is $\overline{X}(2, n) = \overline{M}_{0, n}$ (in particular, (1.14) holds for $\overline{X}(2, n)$) and the equations for the embedding are nice:

1.16. THEOREM. *Let $\kappa := \omega_{\overline{M}_{0, N}}(B)$. Let*

$$N_3 \subset N_4 \subset \cdots \subset N_n = N$$

be a flag of subsets of N , with $|N_j| = j$. There is a canonical identification

$$H^0(\overline{M}_{0, N}, \omega(B)) = \bigotimes_{j=3}^{n-1} V_{N_j}$$

and over the complex numbers a canonical identification

$$H^0(\overline{M}_{0,N}, \kappa) = H^{n-3}(M_{0,N}, \mathbb{C}).$$

κ is very ample, the embedding factors through the Segre embedding

$$\overline{M}_{0,N} \subset \mathbb{P}(V_{N_3}^\vee) \times \mathbb{P}(V_{N_4}^\vee) \cdots \times \mathbb{P}(V_{N_{n-1}}^\vee) \subset \mathbb{P}(H^0(\overline{M}_{0,N}, \kappa)^\vee)$$

and $\overline{M}_{0,N}$ is the scheme theoretic intersection of the Segre embeddings over flags of subsets, i.e.

$$\overline{M}_{0,N} = \bigcap_{N_3 \subset N_4 \cdots \subset N} \mathbb{P}(V_{N_3}^\vee) \times \mathbb{P}(V_{N_4}^\vee) \times \mathbb{P}(V_{N_{n-1}}^\vee) \subset \mathbb{P}(H^0(\overline{M}_{0,N}, \kappa)^\vee).$$

$$\mathrm{Sym}(H^0(\overline{M}_{0,N}, \kappa)) \rightarrow \bigoplus_{n \geq 0} H^0(\overline{M}_{0,N}, \kappa^{\otimes n})$$

is surjective, and the kernel is generated by quadrics, and the syzygies among the quadrics are generated by linear syzygies.

Above V_{N_i} is the standard k -representation of the symmetric group S_{N_i} . The proof of (1.16) will appear elsewhere. (1.16) implies in particular that the compactification $M_{0,n} \subset \overline{M}_{0,n}$ can be recovered in a canonical way from the system of S_i modules $H^{i-3}(M_{0,i}, \mathbb{Z})$ together with the pullback maps between them (for the fibrations given by dropping points), i.e. can be recovered canonically from the Lie operad, see [Ge].

1.17. THE CHOW QUOTIENT OF $G(3, 6)$. We have proven (1.14) also for $\overline{X}(3, 6)$. This space is very interesting. For example: There is a natural map $\overline{X}(r, n+1) \rightarrow \overline{X}(r, n)$ given by dropping the last hyperplane. For $\overline{X}(2, n)$ it is well known that the map is flat, and canonically identified with the universal family (1.4.1). For $\overline{X}(3, 5)$ it is again flat (this fails for $\overline{X}(3, n)$, $n \geq 6$), and a natural universal family. However it is not the family (1.4.1), but rather Lafforgue's analogous family (he defines such a family beginning with any configuration of hyperplanes, see §2) for the configuration of 10 lines which is dual to the configuration of 5 general lines. There are 15 irreducible components of the boundary $B \subset \overline{X}(3, 6)$ which surject onto $\overline{X}(3, 5)$. If we let Γ be their union, then

$$(\overline{X}(3, 6), \Gamma) \rightarrow \overline{X}(3, 5)$$

gives a flat family of pairs, compactifying the family of pairs (S, B) for S a del Pezzo surface of degree 4, and B a union of some of its -1 curves. A detailed study of $\overline{X}(3, 6)$, including proofs of all the claims of this paragraph, will appear elsewhere.

1.18. Using results of [La] we give a cohomological criterion under which $\overline{X}(r, n)$ will be a log minimal model, see (2.21). This we expect will apply in the cases of (1.14). Note the statement $M \subset \overline{M}$ is the log canonical model has two, in general entirely independent, parts: First a singularity statement, (\overline{M}, B) is log canonical (morally, toroidal), and second a positivity statement, $K_{\overline{M}} + B$ is ample. However by (1.4.1) and (2.21) in the case of $X(r, n) \subset \overline{X}(r, n)$ it turns out that whenever the first happens, the second comes for free. See (2.21) for the precise statement.

1.19. MNEV'S UNIVERSALITY THEOREM. The boundary

$$\mathbb{P}(r, n) \setminus \mathbb{P}^\circ(r, n)$$

is a union of $\binom{n}{r}$ Weil divisors. The components have only mild singularities, however they meet in very complicated ways: Let Y be an affine scheme of finite type over $\text{Spec}(\mathbb{Z})$. By [La, 1.8], there are integers n, m and an open subset

$$U \subset Y \times \mathbb{A}^m$$

such that the projection $U \rightarrow Y$ is surjective, and U is isomorphic to a boundary stratum of $\mathbb{P}(3, n)$ (a boundary stratum for a divisorial boundary means the locally closed subset of points which lie in each of a prescribed subset of the irreducible components, but no others).

An analogous statement holds for the boundary of $X(3, n)$, say in any of its smooth GIT quotient compactifications. Our (3.16) suggests the singularities of the pair $(\overline{X}(3, n), B)$ are also in general arbitrary scheme-theoretically. Now by (2.20) and the finite generation conjecture of Mori theory, $X(3, n) \subset \overline{X}_{lc}(3, n)$ will give a canonical compactification in which the boundary has mild (i.e. log canonical) singularities. $\overline{X}(r, n)$ maps to all the GIT quotients. We do not know whether or not this will hold for $\overline{X}_{lc}(3, n)$ – if it does \overline{X}_{lc} would give an absolutely canonical way of (partially) resolving a boundary whose strata include all possible singularities.

Now we return to the Tits building, to give a canonical description of special fibres.

1.20. DEFINITION. We define the *membrane*, $[\mathcal{F}] \subset \mathcal{B}$ to be classes of lattices which have a basis given by scalar multiples of some r elements from \mathcal{F} , or equivalently, such that the limits $\mathcal{F}^{\overline{M}}$ contain a basis of \overline{M} .

1.21. SPECIAL FIBERS. The building \mathcal{B} is a simplicial complex of dimension $r - 1$: We say $[M], [N] \in \mathcal{B}$ are *incident* if we can choose representatives so

$$zM \subset N \subset M$$

(the relation is easily seen to be symmetric). Points $[M_1], \dots, [M_m]$ span an $(m - 1)$ -simplex iff they are pairwise incident, which holds iff we can choose representatives so

$$zM_m = M_0 \subset M_1 \subset \dots \subset M_m.$$

By scaling we can put any of the M_i in the position of M_m but the cyclic ordering among them is intrinsic.

Now take a convex subset

$$Y \subset [\mathcal{F}] \subset \mathcal{B}$$

(not necessarily finite). Canonically associated to each $(m - 1)$ -simplex $\sigma \subset Y$ as above, is a smooth projective variety $\tilde{\mathcal{P}}(\overline{\sigma})$,

$$\tilde{\mathcal{P}}(\overline{\sigma}) = \prod_{m \geq i \geq 1} \tilde{\mathcal{P}}(M_i/M_{i-1})$$

where $\tilde{\mathcal{P}}(M_i/M_{i-1})$ is a certain iterated blowup of the projective space of quotients $\mathcal{P}(M_i/M_{i-1})$ along smooth centers. For precise details see (5.10), (5.16). There are canonical compatible closed embeddings $\tilde{\mathcal{P}}(\overline{\sigma}) \subset \tilde{\mathcal{P}}(\overline{\gamma})$

for simplicies $\gamma \subset \sigma \subset Y$. Finally there is a canonical scheme S_Y , with irreducible components $\tilde{\mathcal{P}}(\overline{M})$, $[M] \in Y$, such that for a subset $\sigma \subset Y$, the $\tilde{\mathcal{P}}(\overline{M})$, $[M] \in \sigma$ have common intersection iff σ is a simplex, and in that case the intersection is $\tilde{\mathcal{P}}(\sigma)$, e.g. $\tilde{\mathcal{P}}(\overline{M})$ and $\tilde{\mathcal{P}}(\overline{N})$ for lattices $[M], [N] \in Y$ meet iff they span a 1-simplex, σ , and in that case they are glued along the common smooth divisor $\tilde{\mathcal{P}}(\overline{\sigma})$. In particular, S_Y has smooth components and normal crossings. When Y is finite, S_Y is the special fiber of \mathbb{S}_Y . It carries Cartier boundary divisors $B_i \subset S_Y$ for each $i \in N$. These are described as follows: B_i has smooth irreducible components, and $\sum B_i$ has normal crossings. B_i has a component on $\tilde{\mathcal{P}}(\overline{M}) \subset S_Y$, $[M] \in Y$ iff the lattice $[M + z^{-1}f_i^M] \in \mathcal{B}$ is not in Y . In this case the component is the strict transform of the hypersurface $\{f_i^M = 0\} \subset \mathcal{P}(\overline{M})$. The limit variety (S, B) (i.e. the fibre of (S, \mathcal{B}) over the image of the closed point of R) is the $K_{S_Y} + B$ minimal model, see (1.24).

1.22. BUBBLE SPACE. For $Y = [\mathcal{F}]$, write $S_\infty = S_Y$. For $r = 2$, S_∞ is the scheme constructed in [Kal] – it is a tree of rational curves with countably many components such that each component intersects at least two others. S_∞ has no boundary, its canonical linear series $|K_{S_\infty}|$ is globally generated and the image of the associated map is again S . More precisely:

1.23. THEOREM. S_∞ has smooth projective components and normal crossings. It carries a natural vector bundle $\Omega^1(\log)$, with determinant ω_{S_∞} . For each finite convex subset

$$\text{Stab} \subset Y \subset [\mathcal{F}]$$

there is a canonical regular surjection

$$p : S_\infty \rightarrow S_Y$$

and a canonical isomorphism

$$p^*(\Omega^1(\log B)) \rightarrow \Omega^1(\log).$$

Given a closed point $x \in S_\infty$ there exists a Y so that p is an isomorphism in a neighborhood of x .

The differential forms $d\log(f/g)$, $f, g \in \mathcal{F}$, induce a canonical inclusion

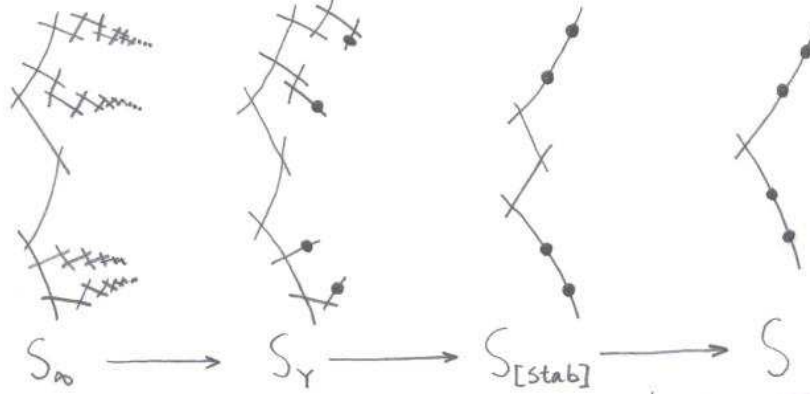
$$V_n \subset H^0(S_\infty, \Omega^1(\log)).$$

These sections generate the bundle. The associated map

$$S_\infty \rightarrow G(r-1, n-1)$$

factors through S_Y and the image is the limit variety S .

The picture illustrates the case $r = 2$:



1.24. THE LIMIT VARIETY (S, B) can also be canonically recovered from the membrane: For each $[M] \in [\mathcal{F}]$, $\tilde{\mathcal{P}}(\overline{M})$ has a canonical normal crossing boundary, B , the union of the divisors $\tilde{\mathcal{P}}(\overline{\sigma}) \subset \tilde{\mathcal{P}}(\overline{M})$ over 1-simplices $[M] \in \sigma \subset [\mathcal{F}]$. The rational differential forms $d\log(f^{\overline{M}}/g^{\overline{M}})$ on $\tilde{\mathcal{P}}(\overline{M})$ have log poles along B , and so define canonical sections of $\Omega^1(\log B)$. These sections generate the bundle. In particular their wedges generate $K_{\tilde{\mathcal{P}}(\overline{M})} + B$.

1.25. DEFINITION. We call a configuration of hyperplanes *GIT stable* if its group of automorphisms is trivial. We call $[M] \in [\mathcal{F}]$ *GIT stable* if the configuration of limiting hyperplanes $\mathcal{F}^{\overline{M}}$ is GIT stable. Of course stable implies GIT stable. For $r \leq 3$ they are the same, see 8.9, but they are in general different. There are only finitely many GIT stable equivalence classes, see 8.5. $[M]$ is GIT stable if and only if $K_{\tilde{\mathcal{P}}(\overline{M})} + B$ is big.

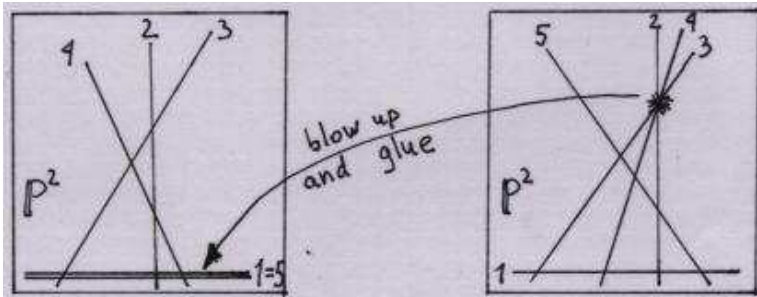
The irreducible components of S are the $(K_{\tilde{\mathcal{P}}(\overline{M})} + B)$ -minimal models of $\tilde{\mathcal{P}}(\overline{M})$ for GIT stable \overline{M} . The minimal model can be constructed as follows: Let $U_{\overline{M}} \subset \mathcal{P}(\overline{M})$ be the complement to the union of hyperplanes. Equivalently, $U_{\overline{M}} = \tilde{\mathcal{P}}(\overline{M}) \setminus B$. The (regular) differential forms $d\log(f^{\overline{M}}/g^{\overline{M}})$ generate the cotangent bundle of $U_{\overline{M}}$, the associated map to $G(r-1, n-1)$ is an immersion, and the corresponding irreducible component of S is the closure of $U_{\overline{M}} \subset G(r-1, n-1)$.

1.26. ILLUSTRATION. Let us look at the first non-trivial example for $r = 3$.

$$\mathcal{F} = \{f_1, f_2, f_3, \quad f_4 = f_1 + f_2 + f_3, \quad f_5 = z^{-1}f_1 + f_2 + f_3\}$$

for f_1, f_2, f_3 the standard (constant) basis of $k^3 \subset K^3$. In this case there are two stable lattices,

$$M_1 = Rf_1 + Rf_2 + Rf_3 \quad \text{and} \quad M_2 = Rz^{-1}f_1 + Rf_2 + Rf_3.$$



The picture illustrates the limit surface (S, B) . Note $zM_2 \subset M_1 \subset M_2$, so the stable lattices in this case form a 1-simplex, σ , in particular are already convex. So we can take $Y = \sigma = \text{Stab}$. The two pictures are the configurations of limit lines. The components of S are $\tilde{\mathcal{P}}(\overline{M}_1) = \mathcal{P}(\overline{M}_1)$, and $\tilde{\mathcal{P}}(\overline{M}_2)$, the blowup of $\mathcal{P}(\overline{M}_2)$ at the intersection point $f_2^{\overline{M}_3} = f_3^{\overline{M}_3} = 0$. The two components are glued along $\tilde{\mathcal{P}}(\overline{\sigma}) = \mathcal{P}(M_2/M_1)$, which embeds in $\tilde{\mathcal{P}}(\overline{M}_1)$ as the line $f_1^{\overline{M}_1} = 0$, and in $\tilde{\mathcal{P}}(\overline{M}_2)$ as the exceptional curve.

Unfortunately we can't draw the membrane: if $n \geq 5$ then no membrane of $\overline{X}(3, n)$ can be embedded in \mathbb{R}^3 without self-intersections.

1.27. RELATION TO TROPICAL ALGEBRAIC GEOMETRY. There are several connections between this work and tropical algebraic geometry: $[\mathcal{F}]$ is naturally homeomorphic to the tropicalisation of the r -dimensional subspace in K^n defined by the rows of the matrix with columns the f_i , see (4.15). Further we observe, (4.16) a natural generalisation of Kapranov's family (1.4.1) which might give information about an arbitrary tropical variety. It may also help explain an interesting correspondence, mysterious to us at present: We have observed that the incident combinatorics of the boundary of $\overline{X}(r, n)$ are encoded in the tropical Grassmannian, introduced in [SS], at least in the cases $\overline{X}(2, n)$ and $\overline{X}(3, 6)$, which are the only cases in which the tropical Grassmannian, or the boundary strata, have been computed.

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§2. VARIOUS TORIC QUOTIENTS OF THE GRASSMANNIAN

2.1. CHOW VARIETY [Ba]. Let $\text{Chow}_{k,d}(\mathbb{P}^n)$ be the set of all k -dimensional algebraic cycles of degree d in \mathbb{P}^n . There is a canonical embedding

$$\text{Chow}_{k,d}(\mathbb{P}^n) \subset \mathbb{P}(\mathcal{V}), \quad X \mapsto R_X,$$

where R_X is a Chow form of X . Here $\mathcal{V} = H^0(G(n-k, n+1), \mathcal{O}(d))^\vee$. The image is Zariski closed, so $\text{Chow}_{k,d}(\mathbb{P}^n)$ is a projective variety with a canonical Chow polarization.

Now if $X \subset \mathbb{P}^n$ is any projective subvariety and $\delta \in H_{2k}(X, \mathbb{Z})$ then the set $\text{Chow}_\delta(X)$ of algebraic cycles in X with the homology class δ is a Zariski closed subset of $\text{Chow}_{k,d}(\mathbb{P}^n)$, where $d \in H_{2k}(\mathbb{P}^n, \mathbb{Z})$ is the image of δ . The resulting structure of the algebraic variety on $\text{Chow}_\delta(X)$ does not depend on the projective embedding.

2.2. CHOW QUOTIENTS [KSZ]. Let H be an algebraic group acting on a projective variety G . Let $G^0 \subset G$ be a (sufficiently) generic open H -invariant

subset. In particular, all orbit closures \overline{Hx} , $x \in G^0$ have the same homology class δ . There is a natural map

$$G^0/H \rightarrow \text{Chow}_\delta(G), \quad x \mapsto \overline{Hx}.$$

The Chow quotient $G//H$ is the closure of the image of this map. There is also a parallel theory of Hilbert quotients $G///H$ when one takes the closure of G^0/H in the Hilbert scheme of G .

2.3. CHOW QUOTIENTS OF PROJECTIVE SPACES [KSZ, GKZ]. Let H be an algebraic torus with the character lattice \mathbb{X} . Let $P \subset \mathbb{X}_{\mathbb{R}}$ be a convex polytope with vertices in \mathbb{X} . We will denote vertices of P by the same letter. Let V be a k -vector space with a basis $\{z_p \mid p \in P\}$. The torus H acts on V by formula $h \cdot z_p = p(h)z_p$. We are going to describe $\mathbb{P}//H$, where $\mathbb{P} = \mathbb{P}(V)$. For any $S \subset \mathbb{P}$, let $\text{Supp}(S) \subset P$ be the set of coordinates that don't vanish on S . Let $\mathbb{P}^0 = \{p \in \mathbb{P} \mid \text{Supp}(p) = P\}$.

Take the big torus $\mathcal{H} = \mathbb{G}_m^P$ with its obvious action on V (so P is identified with the set of “coordinate” characters of \mathcal{H}). We can assume without loss of generality that $H \subset \mathcal{H}$. This is equivalent to $\langle P \rangle_{\mathbb{Z}} = \mathbb{X}$, where for any $S \subset \mathbb{X}$ we denote by $\langle S \rangle_{\mathbb{Z}}$ the minimal sublattice containing S .

Actions of H and \mathcal{H} on \mathbb{P} commute, therefore \mathcal{H} acts on $\mathbb{P}//H$. Moreover, since all points in \mathbb{P}^0 are \mathcal{H} -equivalent, \mathcal{H} has an open orbit $\mathbb{P}^0/H \subset \mathbb{P}//H$. So $\mathbb{P}//H$ is a projective \mathcal{H} -toric variety with the canonical \mathcal{H} -equivariant Chow polarization. By a toric variety we mean a variety with an action of a torus having a dense open orbit. We don't assume that the action is effective or that the variety is normal.

Let $\Psi : \mathbb{P}//_n H \rightarrow \mathbb{P}//H$ be the normalization.

2.4. REMARK. Ψ is bijective on the set of orbits (this is true for any projective toric variety with an equivariant polarization) but (as far as we know) is not always a bijective map. We study this issue in more detail at the end of this section, see (2.25).

The fan $\mathcal{F}(P)$ of $\mathbb{P}//_n H$ can be described as follows. A triangulation T of P (with all vertices in the set of vertices of P) is called coherent if there exists a concave piecewise affine function on P whose domains of affinity are precisely maximal simplices of T . It gives rise to a polyhedral cone $C(T) \subset \mathbb{R}^P$ of the maximal dimension. Namely, $C(T)$ consists of all functions $\psi : P \rightarrow \mathbb{R}$ such that $\psi_T : P \rightarrow \mathbb{R}$ is concave, where ψ_T is given by affinely interpolating ψ inside each simplex of T . Cones $C(T)$ (and their faces) for various T give a complete fan $\mathcal{F}(P)$. Lower-dimensional faces of $\mathcal{F}(P)$ correspond to (coherent) polyhedral decompositions \underline{P} of P . More precisely, $C(\underline{P})$ is the set of concave functions affine on each polytope of \underline{P} .

We will on occasion abuse notation and refer to the collection of maximal dimensional polytopes of a polyhedral decomposition as a polyhedral composition itself.

We have the orbit decomposition

$$\mathbb{P}//H = \bigsqcup_{\underline{P}} (\mathbb{P}//H)_{\underline{P}}$$

(and a similar one for $\mathbb{P} //_n H$) indexed by polyhedral decompositions. A cycle $X \in (\mathbb{P} // H)_{\underline{P}}$ is the union of toric orbits with multiplicities, moreover

$$(2.4.1) \quad X = \sum_{P' \in \underline{P}} m_{P'} X_{P'}, \quad \text{Supp}(X_{P'}) = P', \quad m_{P'} = [\mathbb{X} : \langle P' \rangle_{\mathbb{Z}}].$$

If $m_{P'} = 1$ for any $P' \in \underline{P}$ then we say that X is a *broken toric variety*. If

$$\langle P'' \rangle_{\mathbb{Z}} = \mathbb{X} \cap \langle P'' \rangle_{\mathbb{Q}}$$

for any face P'' of a polytope $P' \in \underline{P}$ then we call \underline{P} unimodular.

2.5. HYPERSYMPLEX. Let $H = \mathbb{G}_m^n$ be the standard torus acting on $\wedge^r k^n$. The weights $e_{i_1} + \dots + e_{i_r} \in \mathbb{R}^n$ are the vertices of the hypersimplex

$$\Delta(r, n) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i = r, 1 \geq x_i \geq 0 \right\}.$$

$\Delta(r, n)$ has $2n$ faces $\{x_i = 0\}$ and $\{x_i = 1\}$. The Plücker embedding $G(r, n) \subset \mathbb{P}(\wedge^r k^n)$ induces a closed embedding $G(r, n) // H \subset \mathbb{P}(\wedge^r k^n) // H$.

2.6. LAFFORGUE'S VARIETY \mathcal{A} . If $x \in G(r, n) \subset \mathbb{P}(\wedge^r k^n)$ then a convex hull of $\text{Supp}(x) \subset \Delta(r, n)$ is always a so-called matroid polytope (for the definition see (3.1)). Lafforgue defines his varieties for an arbitrary fixed matroid polytope P . Let

$$\mathbb{P}^P = \{x \in \mathbb{P} \mid \text{Supp}(x) \subset P\}, \quad \mathbb{P}^{P,0} = \{x \in \mathbb{P}^P \mid \text{Supp}(x) = P\}$$

The locally closed subscheme

$$G^{P,0}(r, n) = G(r, n) \cap \mathbb{P}^{P,0}$$

is called a thin Schubert cell. Of course, $G^{\Delta(r,n),0}(r, n) = G^0(r, n)$. Lafforgue's scheme $\overline{\Omega}^P$ (see 2.8) is a compactification of $G^{P,0}(r, n) // H$. In almost all our applications, $P = \Delta(r, n)$ and so we adopt the following Notational Convention throughout the paper: If we drop the polytope P from notation, it is assumed to be $\Delta(r, n)$, for a pair (r, n) clear from context.

In [La, 2.1] Lafforgue defines a subfan of $\mathcal{F}(P)$ whose cones are in one to one correspondence with matroid decompositions \underline{P} of P (i.e. tilings of P by matroid polytopes). This is a fan because a polyhedral decomposition coarser than a matroid decomposition is a matroid decomposition (moreover, if a convex polytope $Q \subset \Delta(r, n)$ admits a tiling by matroid polytopes then Q itself is a matroid polytope, see [La1]). The associated toric variety is denoted \mathcal{A}^P . Just by definition, \mathcal{A}^P is the toric open subset in the normalization of the Chow quotient:

$$(2.6.1) \quad \mathcal{A}^P \subset \mathbb{P}^P //_n H.$$

Orbits in \mathcal{A}^P correspond to matroid decompositions. Notice that the action of \mathbb{G}_m^P on \mathcal{A}^P is not effective: the kernel $(\mathbb{G}_m^P)_{\emptyset} \subset \mathbb{G}_m^P$ is the subtorus of affine maps $P \rightarrow \mathbb{G}_m$. Let $\mathcal{A}_{\emptyset}^P := \mathbb{G}_m^P / (\mathbb{G}_m^P)_{\emptyset}$.

For any face Q of P , Lafforgue defines a natural *face map* of toric varieties $\mathcal{A}^P \rightarrow \mathcal{A}^Q$. The corresponding map of fans is given by the restriction of piecewise affine functions from P to Q . In particular, the image of the orbit $\mathcal{A}_{\underline{P}}^P$ is $\mathcal{A}_{\underline{P}'}^Q$, where \underline{P}' is the matroid decomposition of Q obtained by intersecting polytopes in \underline{P} with Q .

2.7. LAFFORGUE'S VARIETY $\tilde{\mathcal{A}}$. Lafforgue introduces a second normal toric variety $\tilde{\mathcal{A}}^P$ for the torus $\tilde{\mathcal{A}}_\emptyset^P := \mathbb{G}_m^P/\mathbb{G}_m$ and a map of toric varieties

$$(2.7.1) \quad \tilde{\mathcal{A}}^P \rightarrow \mathcal{A}^P$$

extending the natural quotient map $\tilde{\mathcal{A}}_\emptyset^P \rightarrow \mathcal{A}_\emptyset^P$.

The torus orbits of $\tilde{\mathcal{A}}^P$ are in one to one correspondence with (\underline{P}, P') , for \underline{P} a matroid decomposition, and $P' \in \underline{P}$ one of the matroid polytopes.

By [La, Proposition IV.3], (2.7.1) is projective and flat, with geometrically reduced fibres, and there exists a natural equivariant closed embedding

$$(2.7.2) \quad \tilde{\mathcal{A}}^P \subset \mathcal{A}^P \times \mathbb{P}^P.$$

The fibre of (2.7.1) over a closed point of $\mathcal{A}_\underline{P}^P = (\mathbb{P}^P//_n H)_\underline{P}$ is a broken toric variety (2.4.1) in \mathbb{P}^P . All multiplicities are equal to 1 because of the following fundamental observation [GGMS]:

$$(2.7.3) \quad \text{any matroid decomposition is unimodular.}$$

In fact, (2.7.1) is the pullback of the universal Chow family over the Chow quotient $\mathbb{P}^P//H$ along the map $\mathcal{A}^P \subset \mathbb{P}^P//_n H \rightarrow \mathbb{P}^P//H$.

For each maximal face P' of P , the pair (\emptyset, P') , where \emptyset denotes the trivial decomposition (just P and its faces), corresponds to an irreducible boundary divisor of $\tilde{\mathcal{A}}^P$. Denote the union of these boundary divisors as $\tilde{\mathcal{B}}^P \subset \tilde{\mathcal{A}}^P$. In the case of $P = \Delta(r, n)$ there are $2n$ such boundary divisors, corresponding to the maximal faces $\{x_i = 0\}$, $\{x_i = 1\}$ of $\Delta(r, n)$. We indicate by $\tilde{\mathcal{B}}_i$ the divisor corresponding to $\{x_i = 1\}$. Boundary divisors of $\tilde{\mathcal{A}}^P$ induce boundary divisors B on fibres of (2.7.1) for each maximal face of P . For $P = \Delta(r, n)$ we write B_i for the divisor corresponding to $\tilde{\mathcal{B}}_i$.

2.8. LAFFORGUE'S COMPACTIFICATION $\overline{\Omega}$. Next we consider Lafforgue's main object, $\overline{\Omega}^P$, which we consider only in the case $P = \Delta(r, n)$. We use a different construction from his, as it is a quicker way of describing the scheme structure – $\overline{\Omega}$ is the subscheme of \mathcal{A} over which the fibres of (2.7.1) are contained in $G(r, n)$.

2.9. PROPOSITION. *The Lafforgue space $\overline{\Omega} \subset \mathcal{A}$ is $\varphi^{-1}(\text{Hilb}(G(r, n)))$, where $\varphi : \mathcal{A} \rightarrow \text{Hilb}(\mathbb{P}(\wedge^r(k^n)))$ is a map induced by (2.7.2).*

Proof. As Lafforgue pointed out to us, this follows from [La, 4.4, 4.22]. \square

2.10. STRUCTURE MAP. We have the composition

$$\overline{\Omega} \subset \mathcal{A} \rightarrow \mathcal{A}/\mathcal{A}_\emptyset$$

(where the last map is the stack quotient), which Lafforgue calls the *structure map*. In particular this endows $\overline{\Omega}$ with a stratification by locally closed subschemes, $\overline{\Omega}_\underline{P}$ (the restriction of the corresponding toric stratum of \mathcal{A}), parameterized by matroid decompositions \underline{P} of $\Delta(r, n)$. The stratum for the trivial decomposition, \emptyset (meaning the only polytope is $\Delta(r, n)$) is an open subset

$$\overline{\Omega}_\emptyset = X(r, n) \subset \overline{\Omega}$$

which Lafforgue calls the *main stratum*. Lafforgue proves that $\overline{\Omega}$ is projective, and thus gives a *compactification* of $X(r, n)$ – italics as his space is in general reducible, as we observe in (3.13).

2.11. We denote the closure of $\overline{\Omega}_\emptyset$ in $\overline{\Omega}$ by $\overline{X}_L(r, n)$. There are immersions

$$X(r, n) \subset \overline{X}_L(r, n) \subset \mathcal{A} \subset \mathbb{P} //_n H$$

(the first and last open, the middle one closed) and

$$X(r, n) \subset \overline{X}(r, n) \subset \mathbb{P} // H$$

(open followed by closed). It follows that there exists a finite birational map

$$(2.11.1) \quad \overline{X}_L(r, n) \rightarrow \overline{X}(r, n).$$

In particular, $\overline{X}(r, n)$ and $\overline{X}_L(r, n)$ have the same normalization that we denote by $\tilde{X}(r, n)$.

2.12. TORIC FAMILY. We denote the pullback of $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ to $\overline{\Omega}$ by $\mathcal{T} \rightarrow \overline{\Omega}$ (\mathcal{T} to denote toric). By definitions, $\mathcal{T} \subset \overline{\Omega} \times G(r, n)$.

Kapranov [Ka, 1.5.2] shows that $\overline{X}(r, n)$ is isomorphic to the Hilbert quotient $G(r, n) // H$ and the natural Chow family

$$\mathcal{T} \rightarrow \overline{X}(r, n), \quad \mathcal{T} \subset \overline{X}(r, n) \times G(r, n).$$

is flat. The family $\mathcal{T} \rightarrow \overline{X}_L(r, n)$ is the pullback of $\mathcal{T} \rightarrow \overline{X}(r, n)$ along (2.11.1).

Let $\mathcal{B}, \mathcal{B}_i \subset \mathcal{T}$ be the restrictions of the boundary divisors $\tilde{\mathcal{B}}, \tilde{\mathcal{B}}_i \subset \tilde{\mathcal{A}}$.

2.13. FAMILY OF VISIBLE CONTOURS. Let $G_e(r-1, n-1) \subset G(r, n)$ be the subspace of r -planes containing the fixed vector $e = (1, \dots, 1)$. Kapranov defines the *family of visible contours*

$$\mathcal{S} = \mathcal{T} \cap (\overline{X}(r, n) \times G_e(r-1, n-1)) \subset \overline{X}(r, n) \times G(r, n).$$

Kapranov shows that the family \mathcal{S} is flat, and that the associated map

$$(2.13.1) \quad \overline{X}(r, n) \rightarrow \text{Hilb}(G_e(r-1, n-1))$$

is a closed embedding.

There is a similar family over $\overline{\Omega}$ (Lafforgue calls it $\tilde{\mathbb{P}}(\mathcal{E})$):

2.14. DEFINITION. Let $\mathcal{S} \subset \mathcal{T}$ be the scheme theoretic intersection

$$\mathcal{S} := \mathcal{T} \cap [\overline{\Omega} \times G_e(r-1, n-1)] \subset [\overline{\Omega} \times G(r, n)].$$

H acts on $\tilde{\mathcal{A}}$, trivially on \mathcal{A} and $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ is H equivariant. Thus H acts on \mathcal{T} (and trivially on $\overline{\Omega}$) so that $\mathcal{T} \rightarrow \overline{\Omega}$ is equivariant.

Let $\mathcal{B}, \mathcal{B}_i \subset \mathcal{S}$ indicate the restriction of $\mathcal{B}, \mathcal{B}_i \subset \mathcal{T}$. We note $\mathcal{B} \subset \mathcal{S}$ is the union of \mathcal{B}_i , as the n components of $\mathcal{B} \subset \mathcal{T}$ corresponding to the faces $x_i = 0$ of $\Delta(r, n)$ are easily seen to be disjoint from $G_e(r-1, n-1)$.

The fibres of $(\mathcal{S}, \mathcal{B}) \rightarrow \overline{\Omega}$ have singularities like (or better) than those of $(\mathcal{T}, \mathcal{B})$, as follows from the following transversality result:

2.15. PROPOSITION ([La, pg xv]). *The natural map*

$$\mathcal{S} \rightarrow \mathcal{T} / H$$

to the quotient stack (or equivalently, $\mathcal{S} \times H \rightarrow \mathcal{T}$) is smooth.

Proof. We recall for the readers convenience Lafforgue's elegant construction: Let

$$\mathcal{E} \subset G(r, n) \times \mathbb{A}^n$$

be the universal rank r subbundle, and let $\mathring{\mathcal{E}} \subset \mathcal{E}$ be the inverse image under the second projection of the open subset $H \subset \mathbb{A}^n$ (i.e. the subset with all coordinates non-zero). H obviously acts freely on $\mathring{\mathcal{E}}$ and the quotient is canonically identified with $G_e(r-1, n-1)$. This gives a smooth map

$$G_e(r-1, n-1) = \mathring{\mathcal{E}}/H \rightarrow G(r, n)/H$$

Now for any H equivariant $\mathcal{T} \rightarrow G(r, n)$ the construction pulls back, yielding (2.15). \square

Note in particular that this shows

2.16. COROLLARY. $\mathcal{S} \subset \mathcal{T}$ is regularly embedded, with normal bundle the pullback of the universal quotient bundle of $G_e(r-1, n-1)$.

2.17. FIBERS OF \mathcal{S} . A precise description of the fibres of \mathcal{S} is given in [La, Chapter 5]. Here we recall a few points:

Let $S \subset T$ be a closed fibre of $\mathcal{S} \subset \mathcal{T}$ over a point of $\overline{\Omega}_P$. We have by the above a smooth *structure map* $S \rightarrow T/H$, and so S inherits a stratification from the orbit stratification of T/H , parameterized by $P \in \underline{P}$. In particular the facets (maximal dimensional polytopes) of \underline{P} correspond to irreducible components, and the stratum S_P (which are the points of S that lie only on the irreducible component corresponding to P) is the complement in \mathbb{P}^{r-1} to a GIT stable arrangement (see (1.25)) of n hyperplanes with associated matroid polytope P (see (3.1)). The irreducible component itself is the log canonical compactification of S_P , as follows for example from (2.19) below. For $r = 3$ this compactification is smooth, and described by (8.11).

2.18. LOG STRUCTURES AND TORIC STACKS. For basic properties of log structures and toric stacks we refer to [Ol, §5]. Any log structure we use in this paper will be toric, i.e. the space will come with an evident map to a toric variety and we endow the space with the pullback of the toric log structure on the toric variety. In fact, we do not make any use of the log structure itself, only the bundles of log (and relative log) differentials, all of which will be computed by the following basic operation (our notation is chosen with an eye to its immediate application):

Let $q : \tilde{\mathcal{A}} \rightarrow \mathcal{A}$ be a map of toric varieties so that the map of underlying tori is a surjective homomorphism, with kernel H . We have the smooth map

$$\tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}/H$$

(where the target is the stack quotient), and in particular its relative cotangent bundle, which is canonically identified with a trivial bundle with fibre the dual of the Lie algebra to H . We denote the bundle

$$(2.18.1) \quad \Omega_q^1(\log) = \Omega_{\tilde{\mathcal{A}}/(\tilde{\mathcal{A}}/H)}^1$$

as q is log smooth and this is its bundle of relative log differentials, as follows from [Ol, 5.14] and [Ol1, 3.7].

For a map $\overline{\Omega} \rightarrow \mathcal{A}$, consider the pullback

$$\mathcal{T} := \tilde{\mathcal{A}} \times_{\mathcal{A}} \overline{\Omega} \rightarrow \overline{\Omega}$$

Then (2.18.1) pulls back to the relative cotangent bundle for

$$\mathcal{T} \rightarrow \mathcal{T}/H.$$

$\mathcal{T} \rightarrow \overline{\Omega}$ is again log smooth, with this (trivial) bundle of relative log differentials.

Now suppose $\mathcal{S} \subset \mathcal{T}$ is a closed subscheme, so that the map $\mathcal{S} \times H \rightarrow \mathcal{T}$, or equivalently, $\mathcal{S} \rightarrow \mathcal{T}/H$, is smooth. Then the relative cotangent bundle for

$$\mathcal{S} \rightarrow \mathcal{T}/H$$

is a quotient of the pullback of $\Omega_{\mathcal{T}/(\mathcal{T}/H)}^1$, $p : \mathcal{S} \rightarrow \overline{\Omega}$ is log smooth, with bundle of relative log differentials

$$\Omega_p^1(\log) = \Omega_{\mathcal{S}/(\mathcal{T}/H)}^1.$$

2.19. THEOREM. *The visible contour family $p : \mathcal{S} \rightarrow \overline{\Omega}$ is log smooth. Its bundle of log differentials*

$$\Omega_p^1(\log) = \Omega_{\mathcal{S}/(\mathcal{T}/H)}^1$$

is a quotient of the pullback of $\Omega_{\tilde{\mathcal{A}}/(\tilde{\mathcal{A}}/H)}^1$, which is the trivial bundle $\tilde{\mathcal{A}} \times V_n$. Fibres (S, B) are semi-log canonical, and the restriction of the Plücker polarisation to $S \subset G_e(r-1, n-1)$ is $\mathcal{O}(K_S + B)$.

Proof. Let $(S, B) \subset (T, B)$ be closed fibres of $(\mathcal{S}, \mathcal{B}) \subset (\mathcal{T}, \mathcal{B})$. (T, B) is semi-log canonical, and $\mathcal{O}(K_T + B)$ is canonically trivial, e.g. by [Al, 3.1]. (S, B) is now semi-log canonical by (2.15), and by adjunction $\mathcal{O}(K_S + B)$ is the determinant of its normal bundle, which is the Plücker polarisation by (2.16). The other claims are immediate from (2.15) and the general discussion (2.18). \square

The initial motivation for this paper was the elementary observation:

2.20. PROPOSITION. *$X(r, n)$ is minimal of log general type.*

Proof. We show that the first log canonical map on $X(r, n)$ is a regular immersion. Fixing the first $r+1$ hyperplanes identifies $X(r, n)$ with an open subset of $U^{n-(r+1)}$, where $U \subset \mathbb{P}^{r-1}$ is the complement to B , the union of $r+1$ fixed hyperplanes in linear general position. $K_{\mathbb{P}^{r-1}} + B = \mathcal{O}(1)$, so the first log canonical map on U is just the inclusion $U \subset \mathbb{P}^{r-1}$, in particular an immersion. The result follows easily. \square

We have the following criterion to guarantee

$$X(r, n) \subset \overline{X}_L(r, n)$$

is a log minimal model. Let $T_p(\log)$ be the dual bundle to $\Omega_p^1(\log)$ on \mathcal{S} – i.e. the relative tangent bundle to $\mathcal{S} \rightarrow \mathcal{T}/H$.

2.21. THEOREM. *If $R^2 p_*(T_p(\log))$ vanishes at a point of $\overline{X}_L(r, n) \subset \overline{\Omega}$, then $\overline{\Omega} \rightarrow \mathcal{A}/\mathcal{A}_\emptyset$ is smooth, $\overline{X}_L(r, n) = \overline{\Omega}$, $\overline{\Omega}$ is normal, and the pair $(\overline{X}_L(r, n), B)$ has toroidal singularities, near the point.*

If $R^2 p_*(T_p(\log))$ vanishes identically along $\overline{X}_L(r, n)$, then the sheaf

$$\Omega_{\overline{X}_L(r, n)/k}^1(\log B)$$

(defined in (9.1)) is locally free, globally generated, and its determinant, $\mathcal{O}(K_{\overline{X}_L} + B)$, is globally generated and big. In particular

$$X(r, n) \subset \overline{X}_L(r, n)$$

is a log minimal model.

Proof. By [La, 4.25.ii, 5.15], vanishing of R^2 implies the structure map is smooth. Now suppose the structure map is smooth along $\overline{X}_L(r, n)$. The bundle of log differentials for the toric log structure on a normal toric variety is precisely the bundle (9.1), which implies the analogous statement for $(\overline{X}_L(r, n), B)$. The bundle of differentials is the cotangent bundle of the structure sheaf, and thus a quotient of the cotangent bundle to

$$\tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}/\tilde{\mathcal{A}}_\emptyset$$

which by (2.18) is canonically trivial, whence the global generation. Now $K_{\overline{X}_L(r, n)} + B$ is big by (2.20). \square

2.22. REMARK. If the conditions of the theorem hold, then to show $\overline{X}_L(r, n)$ is the log canonical model, it remains to show $K_{\overline{X}_L} + B$ is ample, not just big and nef. We have proven this for $\overline{X}_L(3, 6)$, by restricting to the boundary. We expect it will hold whenever (2.21) applies, i.e. in the cases of (1.14).

2.23. REMARK. When vanishing holds in (2.21) we have generating global sections of $\Omega_{\overline{X}_L(r, n)/k}^1(\log B)$ which give a map

$$\overline{X}_L(r, n) \rightarrow G\left((r-1)(n-r-1), \binom{n}{r} - n\right).$$

For $r = 2$ the sections give a basis of global sections, and we have checked the map is a closed embedding. It would be interesting to know the defining equations. In this case the log canonical series is very ample, and the embedding factors through this embedding into the Grassmannian. The log canonical embedding itself is quite nice, see (1.16).

2.24. WHEN IS Ψ BIJECTIVE? Here we resume a notation of (2.3) and give a technical condition that implies $\Psi|_{(\mathbb{P}/_n H)_\underline{P}}$ is bijective for a polyhedral decomposition \underline{P} . Until the end of this section, we assume that \underline{P} is unimodular.

The construction is a variation of the Ishida's complex of \mathbb{Z} -modules, see [Oda]. Let \underline{P}^i be the set of i -codimensional faces of polytopes in \underline{P} that do not belong to the boundary ∂P . We fix some orientation of each $Q \in \underline{P}^i$. Let A be an abelian group. Consider the complex $C_{\text{Aff}}^\bullet(\underline{P}, A)$ with $C_{\text{Aff}}^i(\underline{P}, A) = \bigoplus_{Q \in \underline{P}^i} \text{Aff}(Q, A)$, where $\text{Aff}(Q, A)$ is the group of affine maps $Q \rightarrow A$. The differential $d^i : C_{\text{Aff}}^i(\underline{P}, A) \rightarrow C_{\text{Aff}}^{i+1}(\underline{P}, A)$ is a direct sum of differentials $d^{Q, R}$ for $Q \in \underline{P}^i$, $R \in \underline{P}^{i+1}$. If R is not a face of Q then $d^{Q, R} = 0$. Otherwise, $d^{Q, R}$ is the restriction map $\text{Aff}(Q, A) \rightarrow \text{Aff}(R, A)$ taken with a negative sign if the fixed orientation of R is opposite to the orientation

induced from Q . Let $H_{\text{Aff}}^\bullet(\underline{P}, A)$ be the cohomology of $C_{\text{Aff}}^\bullet(\underline{P}, A)$. It is clear that $H_{\text{Aff}}^0(\underline{P}, A)$ is the set of piecewise affine functions $\underline{P} \rightarrow A$.

2.25. PROPOSITION. *If $H_{\text{Aff}}^1(\underline{P}, \mathbb{Z}) = 0$ then $\Psi|_{(\mathbb{P}/\!/_n H)_\underline{P}}$ is bijective.*

Proof. We identify \mathcal{H} with maps $P \rightarrow \mathbb{G}_m$. Elements of \mathcal{H} of order N are maps $P \rightarrow \mu_N$ and any map $a : P \rightarrow \mathbb{Z}$ gives a 1-PS $z \mapsto \{p \mapsto z^{a(p)}\}$.

Let $X \in (\mathbb{P}/\!/H)_\underline{P}$ be as in (2.4.1). Let $x \in \Psi^{-1}(X)$. We claim that $\mathcal{H}x \rightarrow \mathcal{H}X$ is bijective. Since $\mathcal{H}_x \subset \mathcal{H}_X$, it suffices to prove that the stabilizer \mathcal{H}_X is connected.

Let $h \in \mathcal{H}_X$. Then $h \in \mathcal{H}_{X_{P'}}$ for any $P' \in \underline{P}$. But if e is a generic point of $X_{P'}$ then

$$\mathcal{H}_{X_{P'}} = \{h \in \mathcal{H} \mid h \cdot e \in X_{P'}\} = \{h \in \mathcal{H} \mid \exists h_{P'} \in H, h \cdot e = h_{P'} \cdot e\}.$$

It follows that $h(p) = h_{P'}(p)$ and hence h is affine on each P' . We see that $\mathcal{H}_X = H_{\text{Aff}}^0(\underline{P}, \mathbb{G}_m)$.

It is enough to show that any element $h \in \mathcal{H}_X$ of a finite order N embeds in a 1-PS $\gamma \subset \mathcal{H}_X$. So let $h \in H_{\text{Aff}}^0(\underline{P}, \mu_N)$. We have the exact sequence

$$0 \rightarrow C_{\text{Aff}}^\bullet(\underline{P}, \mathbb{Z}) \xrightarrow{\cdot N} C_{\text{Aff}}^\bullet(\underline{P}, \mathbb{Z}) \rightarrow C_{\text{Aff}}^\bullet(\underline{P}, \mu_N) \rightarrow 0.$$

Since $H_{\text{Aff}}^1(\underline{P}, \mathbb{Z}) = 0$, there exists an element of $H_{\text{Aff}}^0(\underline{P}, \mathbb{Z})$ that maps to h . The corresponding 1-PS contains h and belongs to the stabilizer \mathcal{H}_X . \square

2.26. DEFINITION. A decomposition \underline{P} is called *central* if $\underline{P}^0 = \{C, S_1, \dots, S_r\}$, where $S_i \cap S_j \subset \partial P$. We call C the central polytope. Let $U_\underline{P} \subset \mathbb{P}/\!/_n H$ be an affine open toric subset with fan $C(\underline{P})$. It contains $(\mathbb{P}/\!/_n H)_\underline{P}$ as the only closed orbit.

2.27. COROLLARY. *If \underline{P} is unimodular and central then $\Psi(U_\underline{P})$ is quasi-smooth, i.e. $\Psi|_{U_\underline{P}}$ is bijective and $U_\underline{P}$ is smooth.*

Proof. To show that $\Psi|_{U_\underline{P}}$ is bijective, it suffices to prove that $\Psi|_{(\mathbb{P}/\!/_n H)_\underline{P}}$ is bijective. Indeed, other strata in $U_\underline{P}$ correspond to decompositions coarser than \underline{P} , which are automatically unimodular and central, so we can use the same argument. It is clear that $\underline{P}^1 = \{F_1, \dots, F_p\}$ is the set of codimension 1 faces of C that are not on the boundary of P . We want to use (2.25). Let $c \in C_{\text{Aff}}^1(\underline{P}, \mathbb{Z})$, so $c = (f_1, \dots, f_p)$, where f_i is affine on F_i . For each i , we have $F_i = C \cap S_j$ for some j . We can choose $g_j \in \text{Aff}(S_j, \mathbb{Z})$ which restricts on f_i (taking into account the orientation). Then c is equal to the differential of the cochain \tilde{c} , where $\tilde{c}(C) = 0$ and $\tilde{c}(S_i) = g_i$.

For the second statement, we have to show that $C(\underline{P}) = (\mathbb{Z}_{\geq 0})^r$ upto global affine functions. Let $f \in C(\underline{P})$. Then f is a concave locally affine function. So $f - f|_C$ is a concave locally affine function that vanishes on C . Let f_i , $1 \leq i \leq r$, be a primitive (i.e. not divisible by an integer) concave locally affine function that vanishes on $P \setminus S_i$. Then $f - f|_C$ is a linear combination of f_i 's with non-negative coefficients. \square

§3. SINGULARITIES OF $(\overline{X}(r, n), B)$

In this section we prove (1.13). The very simple idea is as follows: The notion of log canonical pair $(\overline{X}, \sum B_i)$ generalises normal crossing. In particular, if all the irreducible components B_i are \mathbb{Q} -Cartier, then log canonical

implies at least that the intersection of the B_i has the expected codimension, see (3.21) below. We prove (1.13) by observing that well known configurations give points of $\overline{X}(r, n)$ lying on too many boundary divisors. The main work is to show that these points are actually in the closure of the generic stratum, and that the boundary divisors are Cartier near these points.

3.1. MATROID POLYTOPES [GGMS]. Let $N = \{1, \dots, n\}$. A *matroid* C is N together with a nonempty family of *independent subsets* of N such that any subset of an independent subset is independent and all maximal independent subsets contained in I have the same number of elements for any $I \subset N$. Maximal independent subsets of N are called *bases* of the matroid. Obviously, a subset of a matroid is a matroid (with induced collection of independent subsets). The *rank* of a matroid is equal to the number of elements in any base. A matroid C of rank r gives rise to a *matroid polytope* $P_C \subset \Delta(r, n)$, a convex hull of vertices $e_{i_1} + \dots + e_{i_r}$ for any base $\{i_1, \dots, i_r\} \subset N$. For a subset $I \subset N$, we write $x_I = \sum_{i \in I} x_i$. We consider x_I as a function on $\Delta(r, n)$, in particular $x_I = r - x_{I^c}$. It is known that P_C is defined by inequalities $x_I \leq \text{rank } I$.

3.2. REALIZABLE MATROIDS. Here is the main example: Let $C = \{L_i\}_{i \in N}$ be a configuration of n hyperplanes in \mathbb{P}^{r-1} . Then independent subsets of the corresponding *realizable matroid* (denoted by the same letter C) are subsets of linearly independent hyperplanes. C has rank r if there is at least one independent r -tuple. Let $X_C(r, n)$ be the corresponding moduli space, i.e. N -tuples of hyperplanes with incidence as specified by C modulo PGL_r . The corresponding matroid polytope P_C has a maximal dimension iff C is GIT stable. By the *multiplicity* of a point $p \in \mathbb{P}^{r-1}$ with respect to C we mean the number of hyperplanes in C that contain p , i.e. the usual geometric $\text{mult}_p C$ if we view C as a divisor in \mathbb{P}^{r-1} .

3.3. GELFAND–MACPHERSON CORRESPONDENCE. Let C be as in (3.2). Consider an $r \times n$ matrix M_C with columns given by equations of hyperplanes of C (defined upto a scalar multiple). The row space of M_C gives a point of $G(r, n)$. Thus $X_C(r, n)$ is identified with the quotient of (the reduction of) a thin Schubert cell $G^{P_C, 0}(r, n)/H$ (see (2.6)). So we see that for any $x \in G(r, n)$, $\text{Supp}(x) \subset \Delta(r, n)$ is a matroid polytope of a realizable matroid, in particular the Lafforgue's stratum $\overline{\Omega}_{\underline{P}}$ (2.10) is empty if a matroid decomposition \underline{P} contains non-realizable matroids.

3.4. DIVISOR B_I . It is easy to see that $\{x_I \leq k\}$ is a matroid polytope for any $0 < k < r$. The corresponding configuration is as follows: the only condition we impose is

$$\text{codim} \bigcap_{i \in I} L_i = k.$$

This polytope has full dimension iff $|I| > k$

It follows that if $|I| > k$ and $|I^c| > r - k$ then there is a matroid decomposition of $\Delta(r, n)$ with two polytopes $\{x_I \geq k\}$ and $\{x_I \leq k\}$. The corresponding stratum of \mathcal{A} is maximal among boundary strata. We denote its closure (and corresponding subschemes of $\overline{\Omega}$, $\overline{X}_L(r, n)$, etc.) by B_I .

An example is shown in (1.26), where $r = 3$, $n = 5$, $k = 1$, $I = \{2, 3, 4\}$, $I^c = \{1, 5\}$. In the configuration with polytope $\{x_{I^c} \leq 1\}$, lines of I^c are

identified and lines of I are generic. In the configuration $\{x_I \leq 2\}$, the lines of I have a common point of incidence, and lines of I^c are generic.

3.5. CENTRAL CONFIGURATIONS AND MATROIDS. Let \mathcal{I} be an index set, and for each $\alpha \in \mathcal{I}$, $I_\alpha \subset N$ a subset, such that $|I_\alpha| \geq r$ and

$$(3.5.1) \quad |I_\alpha \cap I_\beta| \leq r - 2 \quad \text{for } \alpha \neq \beta.$$

Let's call $S \subset N$ independent if $|S| < r$ or $|S| = r$ and $S \not\subset I_\alpha$ for any $\alpha \in \mathcal{I}$.

3.6. PROPOSITION. *This gives a structure of a matroid on N .*

Proof. We only have to check that for any $S \subset N$, all maximal independent subsets in S have the same number of elements. It suffices to prove that if S contains an independent set T , $|T| = r$, then any independent subset $R \subset S$ can be embedded in an independent subset with r elements. We can assume that $|R| = r - 1$. If $R \not\subset I_\alpha$ for any $\alpha \in \mathcal{I}$, then we can just add any element to R . If $R \subset I_\alpha$ for some α then this α is unique by (3.5.1) and we add to R an element of T that is not contained in I_α . \square

We call matroids of this form *central*.

3.7. DEFINITION. We say that C is a *central configuration* if a pair (\mathbb{P}^{r-1}, C) has normal crossings on the complement to a 0-dimensional set. If $r = 3$, it simply means that there are no double lines. Let $\mathcal{I} \subset \mathbb{P}^{r-1}$ be the set of points of multiplicity at least r . Then a matroid of C is a central matroid that corresponds to subsets $I_\alpha \subset N$ of hyperplanes containing $\alpha \in \mathcal{I}$.

A polytope P_C of a central matroid C is given by inequalities $x_{I_\alpha} \leq r - 1$ for all $\alpha \in \mathcal{I}$. Let $P_\alpha \subset \Delta(r, n)$ be the matroid polytope $x_{I_\alpha} \geq r - 1$. Let $\underline{\mathcal{I}} = \{P_C, P_\alpha\}_{\alpha \in \mathcal{I}}$.

3.8. LEMMA. *$\underline{\mathcal{I}}$ is a central decomposition of $\Delta(r, n)$ (see (2.26) for the definition) with central polytope P_C . For each subset $\mathcal{I}' \subset \mathcal{I}$, $\underline{\mathcal{I}}'$ is a matroid decomposition, coarser than $\underline{\mathcal{I}}$ and all matroid decompositions coarser than $\underline{\mathcal{I}}$ occur in this way.*

Proof. To show that $\underline{\mathcal{I}}$ is a central decomposition, it suffices to check that $P_\alpha \cap P_\beta$ is on the boundary for any $\alpha \neq \beta$ (this will imply, in particular, that any interior point of any wall $\{x_{I_\alpha} = r - 1\}$ belongs to exactly two polytopes, P_C and P_α). Assume that $x \in P_\alpha \cap P_\beta$ is an interior point of $\Delta(r, n)$. Then $x_{I_\alpha \cap I_\beta} < r - 2$ by (3.5.1) (otherwise $x_i = 1$ for any $i \in I_\alpha \cap I_\beta$ and therefore x is on the boundary). Therefore, $x_{I_\alpha \setminus I_\beta} = x_{I_\alpha} - x_{I_\alpha \cap I_\beta} > 1$ and $x_{I_\alpha \cap I_\beta} = x_{I_\alpha \setminus I_\beta} + x_{I_\beta} > r$. Contradiction.

Any matroid decomposition coarser than $\underline{\mathcal{I}}$ is obviously central and can be obtained by combining P_C with several P_α 's. This has the same effect as taking these α 's out of \mathcal{I} . \square

3.9. PROPOSITION. *Let $U_{\underline{\mathcal{I}}}$ be the affine open toric subset of \mathcal{A} as in (2.26). Then $U_{\underline{\mathcal{I}}}$ is smooth and bijective to $\Psi(U_{\underline{\mathcal{I}}}) \subset \mathbb{P}(\Lambda^r k^n) // H$. Let $U = U_{\underline{\mathcal{I}}} \cap \overline{\Omega}$. $U \subset \overline{X}_L(r, n)$ maps finitely and homeomorphically onto its image in $\overline{X}(r, n)$.*

Proof. Follows from (2.27). \square

3.10. REMARK. It is possible that $\underline{\mathcal{I}}$ corresponds to a central configuration C but central polytopes of decompositions coarser than $\underline{\mathcal{I}}$ are not realizable. In other words, some multiple points of C may be forced by the set of other multiple points (the reader may wish to examine picture (3.13) from this perspective). All configurations used in the proof of (1.13) are of this sort.

3.11. DEFINITION. Fix a hyperplane $L \subset \mathbb{P}^{r-1}$. For each subset $J \subset N$, $|J| \geq r$, Let Q_J be the moduli space of J -tuples of hyperplanes, L_j , $j \in J$ in \mathbb{P}^{r-1} such that the entire collection of hyperplanes, together with L is in linear general position, modulo automorphism of \mathbb{P}^{r-1} preserving L .

Note Q_J is a smooth variety, of dimension $(r-1)(|J|-r)$. Intersecting with the fixed hyperplane L gives a natural smooth surjection

$$Q_J \rightarrow X(r-1, |J|).$$

3.12. LEMMA. Let C , \mathcal{I} , $\underline{\mathcal{I}}$ be as in (3.7). For each $\alpha \in \mathcal{I}$ we have a natural map $X_C(r, n) \rightarrow X(r-1, |I_\alpha|)$, taking the hyperplanes through a . Let

$$M = \prod_{\alpha \in \mathcal{I}} X(r-1, |I_\alpha|) \quad \text{and} \quad Q = \prod_{\alpha \in \mathcal{I}} Q_{I_\alpha}.$$

There is a natural identification

$$\overline{\Omega}_{\underline{\mathcal{I}}} = X_C(r, n) \times_M Q.$$

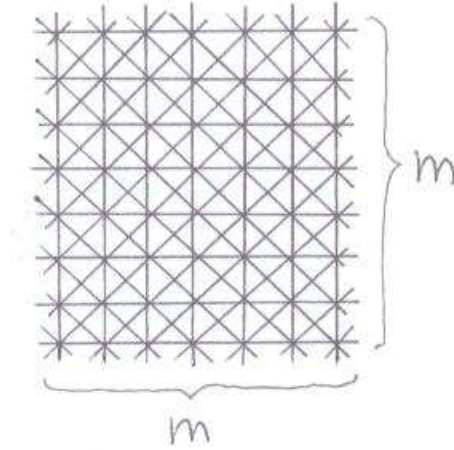
In particular

$$\dim(\overline{\Omega}_{\underline{\mathcal{I}}}) = \dim(X_C(r, n)) + \sum_{\alpha \in \mathcal{I}} (|I_\alpha| - r).$$

Proof. This is immediate from [La, §3.6]. □

Next we demonstrate that Lafforgue's space $\overline{\Omega}$ is reducible:

3.13. PROPOSITION. Let C be the following configuration of $6m-2$ lines in \mathbb{R}^2 :



Let \mathcal{I} be its multiple points, as in (3.7). Then

$$\dim \left(\overline{\Omega}_{\underline{\mathcal{I}}}^{\Delta(3, 6m-2)} \right) \geq m^2$$

and $\overline{\Omega}^{\Delta(3,n)}$ is not irreducible for large n .

Proof. The configuration C has at least m^2 points of multiplicity 4, so the inequality is immediate from (3.12). The final remark follows as the main component $\overline{X}_L(3, 6m - 2)$ of $\overline{\Omega}^{\Delta(3, 6m-2)}$ has dimension $12m - 12$. \square

However, for a large class of central configurations, the stratum $\overline{\Omega}_{\mathcal{I}}$ belongs to the closure of the main stratum:

3.14. LAX CONFIGURATIONS. We say that a central configuration C is *lax* if there is a total ordering on N so that for each $i \in N$, points on L_i of multiplicity *greater* than r with respect to $N_{\leq i}$ are linearly independent. For example, a configuration in (3.13) is not lax for $m \geq 4$.

3.15. THEOREM. *Notation as in (3.7). Assume C is lax.*

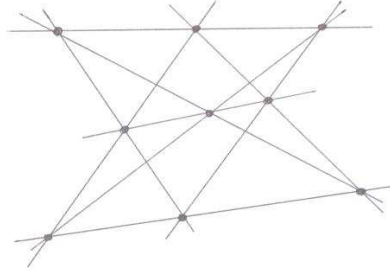
- (3.15.1) *The stratum $\overline{\Omega}_{\mathcal{I}}$ is contained (set theoretically) in $\overline{X}_L(r, n) \subset \overline{\Omega}$.*
- (3.15.2) *Let $U = U_{\mathcal{I}} \cap \overline{\Omega}$, where $U_{\mathcal{I}} \subset \mathcal{A}$ is the smooth toric affine open set of (3.9). Let $\tilde{U} \rightarrow U$ be the normalisation. Then U is an irreducible open factorial subset of $\overline{X}_L(r, n) \subset \overline{\Omega}$, smooth in codimension one. Moreover the boundary strata B_{I_a} are Cartier, generically smooth, and irreducible on U , their union is the boundary, and their scheme-theoretic intersection is the stratum $\overline{\Omega}_{\mathcal{I}}$.*
- (3.15.3) *Let $\tilde{U} \rightarrow U$ be the normalisation, and $\tilde{B} \subset \tilde{U}$ the reduction of the inverse image of B . If $K_{\tilde{U}} + \tilde{B}$ is log canonical at a point in the inverse image of $p \in \overline{\Omega}_{\mathcal{I}}$ then the stratum has pure codimension $|\mathcal{I}|$ in U near p , i.e.*

$$\sum_{\alpha \in \mathcal{I}} (|I_{\alpha}| - r + 1) + \dim X_C(r, n) = n(r - 1) - r^2 + 1$$

near p .

We postpone the proof until the end of this section. First we show that $\tilde{X}(3, n)$ with its boundary fails to be log canonical for $n \geq 9$ (for $n \geq 7$ in characteristic 2) and that $\tilde{X}(4, n)$ is not log canonical for $n \geq 8$.

Proof of (1.13). Consider the Brianchon–Pascal configuration [HC, Do] of 9 lines with $|\mathcal{I}| = 9$ and $|I_{\alpha}| = 3$ for all α :



It is easy to compute that $\dim X_C(3, 9) = 2$. Now apply (3.15): the LHS in (3.15.3) is equal to 11 but the RHS is 10. If $n \geq 10$ add generic lines.

There is an even better configuration of 9 lines with $|Z| = 12$ and $|I_\alpha| = 3$ for all α . It can be obtained as follows: Fix a smooth plane cubic. Every line containing two distinct inflection points contains exactly three. This gives a configuration of 12 lines. Furthermore each inflection point lies on exactly 3 lines, and these are all the intersection points of the configuration. This is the famous Hesse Wendepunkts-configuration [HC, Do]. Let C be the dual configuration. Now apply (3.15): the LHS in (3.15.3) is equal to 12 but the RHS is 10. If $n \geq 10$ add generic lines.

For the characteristic two, use the Fano configuration, [GGMS, 4.5] and argue as above: the LHS in (3.15.3) is equal to 7 but the RHS is 6.

In (4, 8) case, take the configuration of 8 planes in \mathbb{P}^3 given by the faces of the octahedron. There are 12 points of multiplicity 4 (i.e. lying on 4 of the planes), while $\tilde{X}(4, 8)$ is 9 dimensional. If $n \geq 9$ add generic planes. \square

3.16. THEOREM. *The boundary strata of $(\overline{X}(3, n), B)$ for lax configurations have arbitrary singularities, i.e. their reductions give reductions of all possible affine varieties defined over \mathbb{Z} (up to products with \mathbb{A}^1).*

Proof. By (3.15), it suffices to prove that $\overline{\Omega}_{\underline{I}}^{\Delta(3, n)}$ for lax configurations C, \underline{I} , satisfy Mnev's theorem [La, 1.14]. I.e. given affine variety Y over \mathbb{Z} there are integers n, m and an open set $U \subset Y \times \mathbb{A}^m$, with $U \rightarrow Y$ surjective, and a lax configuration C with n lines such that U is isomorphic to the reduction of the Lafforgue stratum $\overline{\Omega}_{\underline{I}}^{\Delta(3, n)}$. One can follow directly the proof of Mnev's theorem: Lafforgue constructs an explicit configuration which encodes the defining equations for Y , and it is easy to check this configuration is lax. The ordering of lines (in Lafforgue's notation) should be as follows: lines $[0, 1_\alpha, P_\alpha, \infty_\alpha]$ and the infinite line should go first (at the end of the process there will be many points of multiplicity > 3 along them), then take all auxilliary lines in the order of their appearance in the Lafforgue's construction. \square

Now we proceed with the proof of (3.15).

3.17. FACE MAPS AND CROSS-RATIOS. The collection of $X(r, n)$ has a hypersimplicial structure: there are obvious maps $B_i : X(r, n) \rightarrow X(r, n - 1)$ (dropping the i -th hyperplane) and $A_i : X(r, n) \rightarrow X(r - 1, n - 1)$ (intersecting with the i -th hyperplane). These maps extend to maps of Chow quotients [Ka, 1.6], and to maps of Lafforgue's varieties $\overline{X}_L \subset \overline{\Omega} \subset \mathcal{A}$, [La, 2.4]. For \mathcal{A} , these maps are just restrictions of face maps (2.6) corresponding to faces $\{x_i = 0\} \simeq \Delta(r, n - 1)$ and $\{x_i = 1\} \simeq \Delta(r - 1, n - 1)$.

In particular, let $V, W \subset N$ be subsets such that $|V| = 4$, $|W| = r - 2$, $V \cap W \neq \emptyset$. Then dropping all hyperplanes not in $V \cup W$ and intersecting with all hyperplanes in W gives cross-ratio maps

$$(3.17.1) \quad CR_{V, W} : X(r, n) \rightarrow X(2, 4) = M_{0, 4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

and

$$CR_{V, W} : \overline{\Omega} \rightarrow \overline{X}(2, 4) = \overline{M}_{0, 4} = \mathbb{P}^1.$$

It follows that $CR_{V, W}(\overline{\Omega}_{\underline{P}}) \subset \mathbb{P}^1 \setminus \{0, 1, \infty\}$ if and only if \underline{P} does not break

$$\Delta_{V, W}(2, 4) = \bigcap_{i \notin V \cup W} \{x_i = 0\} \bigcap \bigcap_{i \in W} \{x_i = 1\}.$$

$\Delta(2, 4)$ is an octahedron and values $\{0, 1, \infty\}$ correspond to three decompositions of $\Delta(2, 4)$ into two pyramids.

To write (3.17.1) as a cross-ratio, let $V = \{i_1 i_2 i_3 i_4\}$ with $i_1 < i_2 < i_3 < i_4$. Let L_1, \dots, L_n be a collection of hyperplanes in $X(r, n)$. Consider an $r \times n$ matrix M with columns given by equations of these hyperplanes. Then

$$CR_{V,W}(L_1, \dots, L_n) = \frac{\text{Det}_{i_1 i_2 W} \text{Det}_{i_3 i_4 W}}{\text{Det}_{i_1 i_3 W} \text{Det}_{i_2 i_4 W}},$$

where each Det_T is an $r \times r$ minor of M with columns given by T .

Let $C = \{L_1, \dots, L_n\}$ be any configuration as in (3.2) and let $x_0 \in G(r, n)$ be a point that corresponds to C under the Gelfand–Macpherson transform. Let $(\mathcal{X}, x_0) \subset G(r, n)$ be a pointed curve such that $\mathcal{X} \cap G^0(r, n) \neq \emptyset$. Let $F : G(r, n)^0 \rightarrow X(r, n)$ be the canonical H -torsor. Then

$$p_0 = \lim_{x \rightarrow x_0} F(x) \in X_L(r, n)$$

belongs to $\overline{\Omega}_{\underline{P}}$, where \underline{P} is a matroid decomposition of $\Delta(r, n)$ containing P_C . Indeed, it is clear that x_0 is contained in the fiber of the universal family (2.7.1) over p_0 , so $P_C = \text{Supp}(x_0)$ is in \underline{P} .

3.18. PROPOSITION. *Let C be central as in (3.7). If*

$$\lim_{x \rightarrow x_0} CR_{V,W}(x) \notin \{0, 1, \infty\}$$

for any $W \subset I_\alpha$, $|V \cap I_\alpha^c| = 1$, $\alpha \in \mathcal{I}$ then $\underline{P} = \underline{\mathcal{I}}$.

Proof. Any decomposition containing P_C is a refinement of $\underline{\mathcal{I}}$. So it remains to prove the following combinatorial statement: any realizable matroid decomposition \underline{P} refining $\underline{\mathcal{I}}$ is equal to $\underline{\mathcal{I}}$ provided that $\underline{P} \cap \Delta(2, 4) = \Delta(2, 4)$ for any face $\Delta(2, 4) \subset \Delta(r, n)$ that belongs to the boundary of some P_α and such that exactly one face of this octahedron $\Delta(2, 4)$ belongs to the wall $x_{I_\alpha} = r - 1$ (this is a condition equivalent to $W \subset I_\alpha$, $|V \cap I_\alpha^c| = 1$).

Restrictions of \underline{P} and $\underline{\mathcal{I}}$ to the faces of $\Delta(r, n)$ have the same form. Also, if $r = 2$ then the claim follows, for example, from the explicit description of matroid decompositions of $\Delta(2, n)$ [Ka, 1.3], so we can argue by induction and it remains to prove the following: any realizable matroid decomposition \underline{P} refining $\underline{\mathcal{I}}$ is equal to $\underline{\mathcal{I}}$ provided that $\underline{P}|_F = \underline{\mathcal{I}}|_F$ for any face $F = \{x_i = 1\}$ of $\Delta(r, n)$, $r > 2$.

Assume, on the contrary, that a certain $P_\alpha \in \underline{\mathcal{I}}$ is broken into pieces. Choose a polytope $Q \subset \underline{P} \cap P_\alpha$ such that the boundary of Q contains the face $F = \{x_l = 1\} \cap P_\alpha$, $l \notin I_\alpha$. A polytope Q is realizable. In the corresponding configuration D , the hyperplane L_l is multiple (of multiplicity $|I_\alpha^c|$). and intersections of hyperplanes L_j , $j \in I_\alpha$ with L_l are in general position (because $F \subset Q$). It follows that D is central (except that L_l is multiple). If $Q \neq P_\alpha$ then there is at least one degeneracy, hyperplanes L_j , $j \in J \subset I_\alpha$, $|J| = r$ pass through a point $\beta \notin L_l$. Since not all hyperplanes L_i , $i \in I_\alpha$ pass through β , there exist indices $k, k' \in J$ and $i \in I_\alpha$ such that a line $\cap_{i \in J \setminus \{k, k'\}} L_i$ intersects L_k and $L_{k'}$ at β , L_l at L_i at two other distinct points. It follows that $\Delta_{\{k, k', l, i\}, J \setminus \{k, k'\}}(2, 4)$ is broken by \underline{P} . \square

Proof of (3.15.1). Let M_C be as in (3.3) for a fixed lax hyperplane arrangement C . Let $Z \subset \overline{\Omega}_{\underline{L}}$ be the fibre over the point of $X_C(r, n)$ given by C , in the product decomposition (3.12).

We consider lines $x_M : \mathbb{A}^1 \rightarrow \mathcal{M}(r, n)$, $x_M(z) = M_C + zM$ for $M \in \mathcal{M}(r, n)$, and the induced regular map (which we abusively denote by the same symbol) $x_M : \mathbb{A}^1 \rightarrow \overline{X}_L(r, n)$. We consider the limit of x_M as $z \rightarrow 0$.

We assume that N has the lax order of (3.14), so for any l , points on L_l of multiplicity greater than r with respect to L_1, \dots, L_{l-1} are linearly independent. Let p_l be the number of these points. A moments thought, and (3.12) yields the equality

$$\sum_{\alpha \in \mathcal{I}} (|I_\alpha| - r) = \sum_i p_i = \dim(Z).$$

We now construct the columns of M . Suppose the first $l-1$ columns of M are already constructed, and consider column l . Let $e_i \in L_l$, $i = 1, \dots, p_l$ be points of multiplicity greater than r . We include these e_i 's in the basis e_1, \dots, e_r and write the l -th column in the dual basis. Let $V_i, W_i \subset N$, $i = 1, \dots, p_n$ be any choice of subsets as in (3.17) such that $V_i = \{i_1, i_2, i_3, l\}$, $|V_i \cap I_{e_i}^c| = 1$, $W_i \subset I_{e_i}$.

3.19. CLAIM. For $i = 1, \dots, p_l$,

$$\lim_{z \rightarrow 0} CR_{V_i, W_i}(x_M(z))$$

does not depend on M_{jl} for $j \neq i$ and depends non-trivially on M_{il} (i.e. we can make this limit any general value by varying M_{il}).

We can assume without loss of generality that $i_1 \notin I_{e_i}$ (otherwise take an appropriate automorphism of a cross-ratio function). Note the claim implies the result: First its clear that any single choice of subsets W, V as in (3.18) can be chosen as W_i, V_i for some i and l . So all these cross ratios are generic (i.e. take on values other than $\{0, 1, \infty\}$) for general M . Now by (3.18) the limit point is in Z . Now by the claim, we can vary $\dim(Z)$ of the cross ratios completely independently by varying M . Since Z is smooth and connected by (3.12) it thus follows that $Z \subset \overline{X}_L(r, n)$ (set theoretically) and so since C was arbitrary this completes the proof.

Let $W := W_i$. Then

$$\lim_{z \rightarrow 0} CR_{V_i, W_i}(x_M(z)) = \lim_{z \rightarrow 0} \frac{\text{Det}_{i_1 i_2 W} \text{Det}_{i_3 l W}}{\text{Det}_{i_1 i_3 W} \text{Det}_{i_2 l W}}$$

Notice that $\lim_{z \rightarrow 0} \text{Det}_{i_1 i_2 W}$ and $\text{Det}_{i_1 i_3 W}$ are not zero - by assumption L_{i_1} does not pass through e_i , but projections of any $r-1$ hyperplanes in I_{e_i} from e_i are linearly independent.

So we have to demonstrate that

$$\lim_{z \rightarrow 0} \frac{\text{Det}_{i_3 l W}}{\text{Det}_{i_2 l W}}$$

does not depend on M_{jl} for $j \neq i$ and depends not trivially on M_{il} . Indeed, the constant terms of $\text{Det}_{i_3 l W}$ and $\text{Det}_{i_2 l W}$ vanish, let's find coefficients at z . The i -th rows of the corresponding submatrices of M_C is trivial, so we can

expand both determinants along this row and get

$$\lim_{z \rightarrow 0} \frac{\text{Det}_{i_3 l W}}{\text{Det}_{i_2 l W}} = \frac{M_{ii_3} R_{ii_3} + M_{il} R_{il} + \dots}{M_{ii_2} R_{ii_2} + M_{il} R_{il} + \dots},$$

where R_{ij} are cofactors of the corresponding submatrices of M_C . These cofactors are not trivial because projections of any $r - 1$ hyperplanes in I_{e_i} from e_i are linearly independent. So we see that the limit indeed does not depend on M_{jl} for $j \neq i$ and is a Möbius function in M_{il} . This function can be made nontrivial by adding an open condition $M_{ii_3} R_{ii_3} \neq M_{ii_2} R_{ii_2}$. \square

3.20. PROPOSITION. *Let C be a lax configuration with multiple points \mathcal{I} . If $|\mathcal{I}| \geq 2$ then $\text{codim}_{\overline{X}_L(r,n)} \overline{\Omega}_{\mathcal{I}} \geq 2$.*

Proof. We proceed by induction on $\sum |I_\alpha|$ using (3.12) and the following observation - a configuration near the GIT-stable configuration is GIT-stable. We will compare quantities $\dim X_C(r, n) + \sum_{\alpha \in \mathcal{I}} (|I_\alpha| - r)$ for various configurations. Since all of them will be GIT stable, we can substitute $X_C(r, n)$ by its PGL_r -torsor $\mathbb{P}_C(r, n)$, the space of all configurations with prescribed multiplicities.

Assume first that there are some points of multiplicity greater than r . Take the last hyperplane L in the lax order that contains such point. Move L a little bit to take it off this point but keep all other points of multiplicity greater than r on L (this is possible because they are linearly independent). If there are no such points, keep some point of multiplicity r (if there are any of them on L). Then the dimension of the configuration space will increase by at least one, the sum $\sum |I_\alpha| - r$ will decrease by at most one and $|\mathcal{I}|$ is still at least 2. At the end, there will be at least two points A, B of multiplicity r and no points of higher multiplicity. Now take a hyperplane through A and move it keeping B if it belongs to this hyperplane. This will increase the dimension of the configuration space by at least one but the result will still not be generic, thus having codimension at least one. \square

Proof of (3.15.2). From (3.12) the generic stratum of B_{I_a} is smooth and connected, and codimension one in U . By (3.20) all other boundary strata of U are lower dimensional. The boundary of U is (by definition) the scheme-theoretic inverse image of the boundary of $U_{\mathcal{I}}$, and so Cartier, and in particular pure codimension one, by (2.27). It follows the B_{I_a} are irreducible, Cartier, and their union is the full boundary. They are generically smooth by (3.12). The proof of (2.20) shows that their complement, the main stratum $X(r, n)$ is isomorphic to an open subset of affine space, and thus has trivial divisor class group. Thus U is factorial. Now it is smooth generically along the Cartier divisors B_{I_a} by (3.12). In the open set $U_{\mathcal{I}} \subset \mathcal{A}$ the stratum $\mathcal{A}_{\mathcal{I}}$ is the scheme-theoretic intersection of the boundary divisors that contain it (this is true in any toric variety). Thus $\overline{\Omega}_{\mathcal{I}}$ is scheme-theoretically the intersection of the boundary divisors of U that contain it. \square

Proof of (3.15.3) now follows from (3.12) and (3.21) below.

3.21. PROPOSITION. *Let X be a normal variety. Let B_i be irreducible Q -Cartier Weil divisors. If $K_X + \sum B_i$ is log canonical, then the intersection*

$$B_1 \cap B_2 \cdots \cap B_n$$

is (either empty or) pure codimension n .

Proof. We can intersect with a general hyperplane to reduce to the case when n is the dimension of X and then apply [FA, 18.22]. \square

§4. THE MEMBRANE

4.1. TITS BUILDINGS. We begin with some background on buildings. For further details see [Mu, §1]. $R = k[[z]]$ is the ring of power series and K is its quotient field. k , as throughout the paper, is an algebraically closed field. Let $V = k^r$ and $V_K = V \otimes_k K$.

We follow [Sp, pg 108] for elementary definitions and properties of simplicial complexes. In particular for us a simplicial complex is a set (the vertex set of the complex) together with a collection of finite subsets called simplices.

4.2. DEFINITION. The *total Grassmannian* $\text{Gr}(V)$ is a simplicial complex of dimension $r - 1$. Its vertices are non-trivial subspaces of V (in particular V itself is a vertex). A collection of distinct subspaces forms a simplex iff they are pairwise incident (i.e. one is contained in the other), from which it easily follows that $m - 1$ simplices correspond to flags of non-trivial subspaces

$$(4.2.1) \quad 0 = U_0 \subset U_1 \subset U_2 \cdots \subset U_m.$$

The compactified *affine Tits building* $\overline{\mathcal{B}}$ is a simplicial complex of dimension $r - 1$, with vertices given by equivalence classes of non-trivial free R -submodules of V_K , where M_1, M_2 are equivalent iff $M_1 = cM_2$ for some $c \in K^*$. A collection of distinct equivalent classes is a simplex iff they are pairwise incident, where $[M_1]$ and $[M_2]$ are called *incident* if after rescaling

$$zM_1 \subset M_2 \subset M_1.$$

Incidence is easily seen to be symmetric.

4.3. LEMMA. *Distinct pairwise incident classes $\Gamma_1, \dots, \Gamma_m \in \overline{\mathcal{B}}$ form a $m - 1$ simplex σ iff after reordering there exist representatives $[M_i] = \Gamma_i$ so that*

$$(4.3.1) \quad zM_m = M_0 \subset M_1 \subset M_2 \cdots \subset M_m.$$

Proof. [Mu, 1.1]. \square

Note that we can rescale so that any M_i in the simplex is in the position of M_m .

The affine Tits building $\mathcal{B} \subset \overline{\mathcal{B}}$ is the full subcomplex of equivalence classes of lattices, i.e. free submodules of rank r (full subcomplex means that a subset of \mathcal{B} forms a simplex iff it does in $\overline{\mathcal{B}}$). As a set, $\mathcal{B} = \text{PGL}_r(K) / \text{PGL}_r(R^*)$.

4.4. STARS. Recall that the *Star* of a simplex σ in a simplicial complex C is a subcomplex

$$\text{Star}_\sigma C = \bigcup_{\sigma \subset \partial \sigma'} \sigma'.$$

Notice that if $\tau \subset \partial \sigma$ then Star_σ is canonically a subcomplex of Star_τ .

4.5. LEMMA. *For any lattice $[\Lambda] \in \mathcal{B}$, there exists a canonical isomorphism*

$$\text{Star}_\Lambda \mathcal{B} \simeq \text{Gr}(\overline{\Lambda}),$$

where $\overline{\Lambda} = \Lambda/z\Lambda$. More generally, for any simplex σ (4.3.1) in \mathcal{B} ,

$$\text{Star}_\sigma \mathcal{B} \simeq \text{Gr}(M_m/M_{m-1}) * \dots * \text{Gr}(M_1/M_0),$$

where $$ denotes the join of simplicial complexes.*

Proof. Indeed, lattices between $z\Lambda$ and Λ are obviously in incidence preserving bijection with subspaces of $\overline{\Lambda}$. Similarly, lattices that fit in the flag σ correspond bijectively to subspaces in one of M_i/M_{i-1} . \square

It is probably worth mentioning that for any simplex σ of $\text{Gr}(V)$ as in (4.2.1), $\text{Star}_\sigma \text{Gr}(V) = \text{Gr}(U_m/U_{m-1}) * \dots * \text{Gr}(U_1/U_0)$.

4.6. RETRACTIONS. For a lattice Λ , and a non-trivial subset $\Theta \subset V_K$ (e.g. an R -submodule or an element) we can find unique $a > 0$ so that $z^a\Theta \subset \Lambda$, $z^a\Theta \not\subset z\Lambda$. We define $\Theta^\Lambda := z^a\Theta \subset \Lambda$. We denote

$$R_\Lambda : \overline{\mathcal{B}} \rightarrow \text{Star}_\Lambda \mathcal{B}$$

the map that sends a submodule M to $M^\Lambda + z\Lambda$.

4.7. LEMMA. R_Λ is a retraction map of simplicial complexes.

Proof. It's clear the map preserves incidence and is identity on $\text{Star}_\Lambda \mathcal{B}$. \square

More generally, for any simplex (4.3.1) in \mathcal{B} there is a retraction

$$R_\sigma : \overline{\mathcal{B}} \rightarrow \text{Star}_\sigma \mathcal{B}$$

defined as follows: $R_\sigma(M) = M^{\Lambda_k} + \Lambda_i$, where i is maximal such that $M^{\Lambda_k} \not\subset \Lambda_i$. We denote by Res_Λ and Res_σ compositions of R_Λ and R_σ with isomorphisms of (4.5).

4.8. CONVEX STRUCTURE. A subset of $\overline{\mathcal{B}}$ is called *convex* if it is closed under finite R -module sums. For a collection of free R -submodules $\{M_\alpha\}$ their *convex hull* in $\overline{\mathcal{B}}$, denoted $[M_\alpha]$ is the subcomplex with vertices (with representatives) of form $\sum c_\alpha M_\alpha$, $c_\alpha \in K$. This is obviously the smallest convex subset that contains all the $[M_\alpha]$.

A subset of $\text{Gr}(V)$ is called *convex* if it is closed under finite sums (of subspaces). We write $[U_\alpha]$ for the convex hull of subspaces $\{U_\alpha\} \subset \text{Gr}(V)$.

It is clear that stars of simplices in \mathcal{B} and $\text{Gr}(V)$ are convex and that isomorphisms (4.5) preserve the convex structure.

4.9. EXAMPLE. For $T = \{g_1, \dots, g_r\}$ a basis of V_K , the convex hull $[T]$ is called an *apartment*. It is the set of equivalence classes $[M]$ such that M has an R -basis c_1g_1, \dots, c_rg_r for some $c_i \in K$.

Retractions commute with convex hulls:

4.10. PROPOSITION. *Let $\{M_\alpha\} \subset \overline{\mathcal{B}}$ be a subset and $\sigma \subset \mathcal{B}$ a simplex. Then*

$$[R_\sigma M_\alpha] = R_\sigma [M_\alpha].$$

If $\sigma \subset [M_\alpha]$ then both sides are also equal to $\text{Star}_\sigma [M_\alpha]$.

Proof. We leave it as an exercise to the reader. \square

4.11. LEMMA ([Fa]). *The convex hull of a finite subset of \mathcal{B} is finite.*

4.12. MEMBRANE. Let

$$\mathcal{F} := \{Rf_1, \dots, Rf_n\}$$

be a collection of n rank 1 submodules, such that $[f_1, \dots, f_n] = V_K$. The convex hull $[\mathcal{F}] \subset \overline{\mathcal{B}}$ we call the *membrane*.

4.13. LEMMA. $[\mathcal{F}]$ is the union of apartments $[T]$ for subsets $T \subset \mathcal{F}$, $|T| = r$.

Proof. This is immediate from the definitions, and Nakayama's lemma. \square

4.14. MEMBRANES AS TROPICAL SUBSPACES. We begin by recalling the construction of the tropical variety (also called a non-Archimedean amoeba or a Bieri–Groves set), see [SS] for details. Let $H = \mathbb{G}_m^n$ be an algebraic torus. Let \overline{K} be the field of generalized Puiseux series $\sum_{\alpha \in I \subset \mathbb{R}} c_\alpha z^\alpha$, where

I is a locally finite subset of \mathbb{R} , bounded below (and is allowed to vary with the series). There is an evaluation map

$$\text{ord} : H(\overline{K}) \rightarrow H(\overline{K})/H(\overline{R}) = \mathbb{R}^n$$

where $\overline{R} \subset \overline{K}$ is the subring of series for which $I \subset \mathbb{R}_{\geq 0}$. For any subvariety $Z \subset H$, $\text{ord}(Z)$ is called the *tropicalisation* of Z . It is a polyhedral complex of dimension $\dim Z$. If Z is invariant under dilations then $\text{ord}(Z)$ is invariant under diagonal translations and we consider

$$\text{Ord}(Z) = \text{ord}(Z) \mod \mathbb{R}(1, \dots, 1).$$

Let $\mathcal{F} := \{Rf_1, \dots, Rf_n\}$ be as in (4.12). Consider the map

$$\Phi : V_K^\vee \rightarrow K^n, \quad F \mapsto (F(f_1), \dots, F(f_n)),$$

Let $Z = \Phi(V_K^\vee) \cap H$. Then Z is of course the intersection with H of the r -plane spanned by the rows of the $r \times n$ matrix with columns given by f_i 's. Its tropicalisation $\text{Ord}(Z) \subset \mathbb{R}^{n-1}$ is called a *tropical projective subspace*.

For any simplicial complex C , we denote by $|C|$ the corresponding topological space (obtained by gluing physical simplices). Recall that $|\mathcal{B}|$ can be identified with the space of equivalence classes of additive norms on V_K , where an additive norm N is a map $V_K(K) \rightarrow \mathbb{R} \cup \{\infty\}$ such that

$$N(cv) = \text{ord}(c) + N(v) \quad \text{for any } c \in K, v \in V_K(K),$$

$$N(u+v) \geq \min(N(u), N(v)) \quad \text{for any } u, v \in V_K(K),$$

and

$$N(u) = \infty \quad \text{iff } u = 0.$$

Two additive norms are equivalent if they differ by a constant. For a norm N let $\tilde{\Psi}(N) = (N(f_1), \dots, N(f_n)) \in \mathbb{R}^n$. Now consider

$$\Psi : |\mathcal{B}| \rightarrow \mathbb{R}^{n-1}, \quad \Psi([N]) = \tilde{\Psi}(N) \mod \mathbb{R}(1, \dots, 1).$$

The map is continuous because the topology on $|\mathcal{B}|$ is exactly the topology of point-wise convergence of norms. The following theorem is our version of the tropical Gelfand–Macpherson transform.

4.15. THEOREM. Ψ induces a homeomorphism $||[\mathcal{F}]| \simeq \text{Ord}(Z)$.

Proof. Let Ω be the Drinfeld's symmetric domain – the complement to the union of all K -rational hyperplanes in $V_K^\vee(\overline{K})$. There is a surjection [Dr]

$$D : \Omega \rightarrow |\mathcal{B}|, \quad F \mapsto [v \rightarrow \text{ord } F(v)] \quad \text{for } v \in V_K(K),$$

here we interpret $|\mathcal{B}|$ as the set of equivalence classes of norms. The following diagram is obviously commutative:

$$\begin{array}{ccc} \Omega & \xrightarrow{D} & |\mathcal{B}| \\ \downarrow \Phi & & \downarrow \Psi \\ H(\overline{K}) & \xrightarrow{\text{Ord}} & \mathbb{R}^{n-1} \end{array}$$

It follows that $\text{Im}(\Psi) \subset \text{Ord}(Z)$.

For any lattice $\Lambda \in \mathcal{B}$, the corresponding norm N_Λ is as follows:

$$N_\Lambda(v) = \{-a \mid z^a v \in \Lambda \setminus z\Lambda\} \in \mathbb{Z}.$$

In particular, $\Psi(\mathcal{B}) \subset \mathbb{Z}^{n-1}$. Also, it follows easily from definitions that Ψ is affine on simplices of $|\mathcal{B}|$ and uniformly continuous. Since $\text{Ord}(Z)$ is a polyhedral complex, it remains to check that for any \mathbb{Q} -point of $\text{Ord}(Z)$, there exists a unique \mathbb{Q} -point of $|\mathcal{F}|$ that maps onto it (a \mathbb{Q} -point of $|\mathcal{B}|$ means a point of some simplex with rational barycentric coordinates). Now we can pass from K to Puiseux series $k[[z^{1/m}]]$ with sufficiently large m (this does not change neither $\text{Ord}(Z)$ nor $|\mathcal{F}|$, see also (6.4) for another version of this baricentric trick) and it remains to check the latter statement for \mathbb{Z} -points. Substituting f_i 's by $z^{a_i} f_i$'s, we can assume that this point is $O = (0, \dots, 0)$. Now we claim that if $O \in \text{ord}(Z)$ then $\tilde{\Psi}(\Lambda) = O$ for $[\Lambda] \in [\mathcal{F}]$ if and only if $\Lambda = Rf_1 + \dots + Rf_n$.

Suppose $\tilde{\Psi}(Rf_1 + \dots + Rf_n) \neq O$, i.e. $f_j \in z(Rf_1 + \dots + Rf_n)$ for some j . By Nakayama's lemma, we can assume without loss of generality that $Rf_1 + \dots + Rf_n = Rf_1 + \dots + Rf_r$. Therefore, $f_j \in z(Rf_1 + \dots + Rf_r)$. But then for any $F \in V_K^\vee(\overline{K})$, if $\text{ord } F(f_i) = 0$ for $i \leq r$ then $\text{ord } F(f_j) > 0$. But this contradicts $O \in \text{ord}(Z)$.

Now take any lattice $\Lambda = Rz^{a_1} f_1 + \dots + Rz^{a_n} f_n$. We can assume without loss of generality that $\Lambda = Rz^{a_1} f_1 + \dots + Rz^{a_r} f_r$. Let $\tilde{\Psi}(\Lambda) = O$. Then $f_i \in \Lambda$ for any i , therefore, $\Lambda = Rf_1 + \dots + Rf_n$. \square

4.16. A GENERALIZATION OF THE VISIBLE CONTOUR FAMILY. Let

$$H = \mathbb{G}_m^{n-1}(K) \subset \mathbb{P}^{n-1}(K).$$

Start with any subvariety $Z \subset H$. Consider the point

$$[\overline{Z}] \in \text{Hilb}_{\mathbb{P}^{n-1}}(K).$$

Assume for simplicity that no element of H preserves Z . We consider the orbit closure $\overline{H \cdot [\overline{Z}]} \subset \text{Hilb}_{\mathbb{P}^{n-1}}(K)$, a toric variety for H , which gives us a K -point

$$[\overline{H \cdot [\overline{Z}]}] \in \text{Hilb}(\text{Hilb}_{\mathbb{P}^{n-1}}),$$

which we can think of as a one parameter family of toric varieties over k . We can then let z approach zero and degenerate the toric varieties to a *broken* toric variety in $\text{Hilb}_{\mathbb{P}^{n-1}}$. Note if we start with \overline{Z} a general $r-1$ plane, then $Z \subset \overline{Z}$ is the complement of n general hyperplanes in \mathbb{P}^{r-1} , the corresponding component of $\text{Hilb}_{\mathbb{P}^{n-1}}$ is $G(r, n)$, and the degeneration

takes place in Kapranov's family of *broken* toric varieties, $\mathcal{T} \rightarrow G(r, n)/H$ of (2.12). The visible contour family also generalizes: Since $Z \subset H$ we can embed Z in $H \cdot [\overline{Z}]$ by $t \rightarrow t^{-1} \cdot [\overline{Z}]$, so the image has the strange expression $Z^{-1} \cdot [\overline{Z}]$. We can then consider

$$\left[\overline{Z^{-1} \cdot [\overline{Z}]} \right] \in \text{Hilb}(\text{Hilb}_{\mathbb{P}^{n-1}})$$

Note $Z^{-1} \cdot [\overline{Z}] \subset \text{Hilb}_{\mathbb{P}^{n-1}}$ is precisely the set of points which (thought of as subschemes of \mathbb{P}^{n-1}) contain the point $(1, \dots, 1)$, so in the case of a general linear space \overline{Z} this is precisely Kapranov's visible contour, (2.13).

In the linear case the broken toric variety is described by a matroid decomposition of $\Delta(r, n)$ which reflects the combinatorics of the simplicial complex $[\mathcal{F}]$ and so by (4.15), of the tropical variety $\text{Ord}(\overline{Z})$. It is natural to wonder if this holds in general.

§5. DELIGNE SCHEMES

For convex

$$\text{Stab} \subset Y \subset \mathcal{F}$$

and $p : \mathbb{S}_Y \rightarrow \text{Spec}(R)$ the Deligne scheme, we have by (5.26) the natural vector bundle $\Omega_p^1(\log \mathbb{B})$, see §9.

5.1. Now we turn to the proofs of (1.9) and (1.21)–(1.24). We follow the notation of the introduction and §4. Here we prove the pair $(\mathbb{S}_Y, S_Y + \mathbb{B})$ of (1.9) has normal crossings, (5.26). Global generation is considered in §6.

5.2. DELIGNE FUNCTOR [Fa]. Let $Y \subset \mathcal{B}$ be a finite set. A *Deligne functor* \mathbb{S}_Y is a functor from R -schemes to sets, a T -valued point q of which consists of a collection of equivalence classes of line bundle quotients

$$q_M : M_T \twoheadrightarrow L(M_T)$$

for each lattice $[M] \in Y$, where $M_T := T \times_R M$, where two quotients are equivalent if they have the same kernel, satisfying the compatibility requirements:

- For each inclusion $i : N \hookrightarrow M$, there is a commutative diagram

$$\begin{array}{ccc} N_T & \xrightarrow{q_N} & L(N_T) \\ i_T \downarrow & & \downarrow \\ M_T & \xrightarrow{q_M} & L(M_T) \end{array}$$

- Multiplication by $c \in K^*$ gives an isomorphism

$$\ker q_M \xrightarrow{\cdot c} \ker q_{cM}$$

It is clear from this definition that \mathbb{S}_Y is represented by a closed subscheme

$$\mathbb{S}_Y \subset \prod_{[M] \in Y} \mathcal{P}(M),$$

$(\mathbb{S}_Y)_K = \mathcal{P}(V_K)$, and \mathbb{S}_Y contains the Mustafin's join (1.8).

5.3. THEOREM [Fa]. *Assume Y is finite and convex. Then \mathbb{S}_Y is smooth and irreducible (in particular it is isomorphic to the Mustafin's join). Its special fiber $S_Y = (\mathbb{S}_Y)_k$ has normal crossings.*

We begin by explaining Faltings' proof of (5.3), recalling and expanding upon the three paragraphs of [Fa, pg. 167]. This is the substance of (5.4)–(5.21). For this $Y \subset \mathcal{B}$ is an arbitrary finite convex subset. Beginning with (5.22) our treatment diverges from [Fa]. We specialize to convex subsets $Y \subset [\mathcal{F}]$ as in (1.9) and consider singularities of the natural boundary.

5.4. MAXIMAL LATTICES [Fa]. Consider a k -point of \mathbb{S}_Y , i.e. a compatible family of one dimensional k -vector space quotients

$$q_M : \overline{M} \rightarrow L(\overline{M}), \quad [M] \in Y,$$

where

$$\overline{M} = M/zM = M \otimes_R k.$$

This gives a partial order on Y : $[N] \leq_q [M]$ iff the composition

$$\overline{N} = \overline{N^M} \rightarrow \overline{M} \rightarrow L(\overline{M})$$

induced by inclusion $N^M \subset M$ is surjective, and thus by compatibility, canonically identified with q_N . In this case we also say that q_M does not vanish on N .

A lattice $[M] \in Y$ is called *maximal* for q if it is maximal with respect to the order \leq_q . In other words, q_N vanishes on M for any $[N] \in Y$, $[N] \neq [M]$. Since Y is finite, it follows that for each $[N] \in Y$ there exists a maximal lattice $[M] \in Y$ such that $[N] \leq_q [M]$.

5.5. LEMMA [Fa]. *Maximal lattices are pairwise incident.*

It follows by (4.3) that maximal lattices form a simplex σ (4.3.1).

5.6. COROLLARY. *A k -point of \mathbb{S}_Y with a simplex of maximal lattices σ is equivalent to a collection of hyperplanes*

$$H_i \subset M_i/M_{i-1}, \quad m \geq i \geq 1,$$

which do not contain $\text{Res}_\sigma[M]$ for any $[M] \in Y$.

Proof. Choose a k -point q of \mathbb{S}_Y with a simplex of maximal lattices σ . Let $[M] \in Y$ and let i be such that $\text{Res}_\sigma[M] \in \text{Gr}(M_i/M_{i-1})$. Rotating M_i 's if necessary, we can assume that $i = 1$. Let M_i be a maximal lattice such that $[M] \leq_q [M_i]$. Then q_{M_i} does not vanish on M , and therefore does not vanish on M_1 . But M_1 is maximal, so $i = 1$. It follows that hyperplanes $H_i = (\ker q_{M_i})/M_{i-1}$ don't contain $\text{Res}_\sigma[M]$ for any $[M] \in Y$. And it is clear that these hyperplanes determine q . \square

5.7. DEFINITION. Let $[M] \in Y$. We let $\tilde{\mathcal{P}}(\overline{M}) \subset \mathbb{S}_Y$ be the subfunctor of compatible quotients such that for each $[N] \in Y$, $N = N^M$, the quotient $q_N : N_T \rightarrow L(N_T)$ vanishes on $(N \cap zM)_T \subset N_T$.

It's clear $\tilde{\mathcal{P}}(\overline{M})$ is represented by a closed subscheme of \mathbb{S}_Y .

5.8. LEMMA. *The k -points of $\tilde{\mathcal{P}}(\overline{M}) \subset \mathbb{S}_Y$ are precisely the set of k -points of \mathbb{S}_Y for which M is a maximal lattice.*

Proof. Consider a k -point q of $\tilde{\mathcal{P}}(\overline{M})$. Suppose $M \subsetneq N$, $[N] \in Y$. Then $N^M = z^k N$, for some $k > 0$. $q_{z^k N}$ vanishes on $z^k N \cap zM$, by the definition of $\tilde{\mathcal{P}}(\overline{M})$. So by compatibility of quotients with scaling q_N vanishes on $N \cap z^{1-k} M$, which contains M . Thus M is a maximal lattice.

Conversely, suppose M is maximal for a k -point q . Take $[N] \in Y$ such that $N = N^M$. By maximality $q_{z^{-1}N+M}$ vanishes on M , thus by compatibility, q_{N+zM} vanishes on zM , thus, again by compatibility, q_N vanishes on $N \cap zM$. So the point lies in $\tilde{\mathcal{P}}(\overline{M})$. \square

5.9. DEFINITION. Let V be a finite-dimensional k -vector space, and let $\mathcal{W} \subset \text{Gr}(V)$ be a finite convex collection of subspaces that includes V . Let $\text{BL}(\mathcal{P}(V), \mathcal{W})$ be the functor from k -schemes to sets which assigns to each T the collection of line bundle quotients $W_T \twoheadrightarrow L(W_T)$, $W \in \mathcal{W}$, W_T the pullback, compatible with the inclusion maps between the W , i.e. the composition

$$A_T \rightarrow B_T \xrightarrow{q_B} L(B_T)$$

factors through $q_A : A_T \twoheadrightarrow L(A_T)$ for $A \subset B$, $A, B \in \mathcal{W}$.

5.10. PROPOSITION. *There is a canonical identification*

$$\tilde{\mathcal{P}}(\overline{M}) = \text{BL}(\mathcal{P}(\overline{M}), \text{Res}_M(Y)).$$

Proof. Immediate from the definitions. \square

5.11. PROPOSITION. $\text{BL}(\mathcal{P}(V), \mathcal{W})$ is represented by the closure of the graph of the product of canonical rational maps $\mathcal{P}(V) \dashrightarrow \mathcal{P}(W)$, $W \in \mathcal{W}$. Furthermore $\text{BL}(\mathcal{P}(V), \mathcal{W})$ is smooth.

Proof. We induct on the number of subspaces in \mathcal{W} . When $\mathcal{W} = \{V\}$ the result is obvious. In any case it is clear the functor is represented by a certain closed subscheme

$$X \subset \prod_{W \in \mathcal{W}} \mathcal{P}(W).$$

Let $\mathcal{P}^0(V) \subset \mathcal{P}(V)$ be an open subset of quotients that don't vanish on any $W \in \mathcal{W}$. Then $\mathcal{P}^0(V)$ is an open subset of X , its closure X' in X is the closure of the graph in the statement.

Take a fixed closed point q^0 . We will show that $X = X'$ near q^0 . Let $W \in \mathcal{W}$ be a maximal subspace such that q_V^0 vanishes on W . If there are none then $q^0 \in \mathcal{P}^0(V)$ and so $X = X'$ near q^0 .

Let $D \subset X$ be the subscheme of compatible quotients so that q_V vanishes on W , let $D^0 \subset D$ be a sufficiently small neighbourhood of q^0 . Let $\mathcal{W}_W \subset \mathcal{W}$ be those subspaces contained in W , clearly \mathcal{W}_W is convex. Take any $E \in \mathcal{W}$ and $q \in D^0$. If $E \notin \mathcal{W}_W$, then q_V^0 (and hence q_V) does not vanish on E , from which it follows that $E \rightarrow L(V)$ is surjective, and thus identified with q_E . It follows easily that D^0 is represented by an open subset of $\text{BL}(\mathcal{P}(W), \mathcal{W}_W) \times \mathcal{P}(V/W)$. In particular, by induction, D^0 is connected and smooth of dimension $\dim V - 2$.

Claim: $D^0 \subset X'$. As D^0 is integral it's enough to check this on some open subset of D^0 . We consider the open subset where q_W does not vanish on any $E \in \mathcal{W}_W$, and q_V does not vanish on any $E \notin \mathcal{W}_W$. This is naturally identified with an open subset for $\mathcal{W} = \{V, W\}$, and so we reduce to this case. But in this case it is easy to see that $X = X'$ is the blowup of $\mathcal{P}(V)$ along $\mathcal{P}(V/W)$ and so obviously $D^0 \subset X'$.

By the Claim X has dimension at least $\dim V - 1$ along D^0 . $D \subset X$ is locally principal, defined by the vanishing of a map between the universal

quotient line bundles for W and V . It follows from (5.12) below that X is smooth, and equal to X' along D^0 . \square

The following is well known:

5.12. LEMMA. *Let (A, m) be a local Noetherian ring of Krull dimension at least d . Assume A/f is regular of dimension at most $d-1$ for $f \in m$. Then A is regular of dimension d .*

Proof.

$$\dim m/m^2 \leq 1 + \dim m/(m^2 + f) = 1 + \dim A/f = d \leq \dim A \leq \dim m/m^2.$$

\square

5.13. DEFINITION. Define the *depth* of $W \in \mathcal{W}$ to be the largest $d \geq 0$ so that there is a proper flag

$$W = W_0 \subset W_1 \cdots \subset W_d = V$$

with $W_i \in \mathcal{W}$. Let $\mathcal{W}_{\leq m} \subset \mathcal{W}$ be the subset of subspaces of depth at most m . Let $\mathcal{W}_m \subset \mathcal{W}_{\leq m}$ be the subset of subspaces of depth exactly m .

Notice that $\mathrm{BL}(\mathcal{P}(V), \mathcal{W}_0) = \mathcal{P}(V)$ and $\mathrm{BL}(\mathcal{P}(V), \mathcal{W}_{\leq N}) = \mathrm{BL}(\mathcal{P}(V), \mathcal{W})$ for $N \gg 0$. Thus the next proposition shows that the canonical map $\mathrm{BL}(\mathcal{P}(V), \mathcal{W}) \rightarrow \mathcal{P}(V)$ is an iterated blowup along smooth centers.

5.14. PROPOSITION. *A forgetful functor*

$$(5.14.1) \quad p : \mathrm{BL}(\mathcal{P}(V), \mathcal{W}_{\leq m+1}) \rightarrow \mathrm{BL}(\mathcal{P}(V), \mathcal{W}_{\leq m})$$

is represented by the blowup along the union of the strict transforms of $\mathcal{P}(V/W) \subset \mathcal{P}(V)$ for $W \in \mathcal{W}_{m+1}$ (which are pairwise disjoint).

Proof. Let $W \in \mathcal{W}_{m+1}$. We claim that the strict transform of $\mathcal{P}(V/W)$ represents the subfunctor X_W of $\mathrm{BL}(\mathcal{P}(V), \mathcal{W}_{\leq m})$ of compatible quotients such that q_E vanishes on $E \cap W$ for all $E \in \mathcal{W}_{\leq m}$. This subfunctor is naturally identified with $\mathrm{BL}(\mathcal{P}(V/W), \mathcal{W}_{\leq m}^W)$, where $\mathcal{W}_{\leq m}^W$ is the (obviously convex) collection of subspaces $(E + W)/W \subset V/W$, for $E \in \mathcal{W}_{\leq m}$. By (5.11), it is smooth and connected, and thus the strict transform. For disjointness: if $W', W'' \in \mathcal{W}_{m+1}$ then $\tilde{W} := W' + W'' \in \mathcal{W}_{\leq m}$ and it is not possible for $q_{\tilde{W}}$ to vanish both on W' and on W'' , thus the strict transforms are disjoint.

The map (5.14.1) is obviously an isomorphism outside the union of subfunctors X_W . Take $W \in \mathcal{W}_{m+1}$. The inverse image

$$p^{-1}(X_W) \subset \mathrm{BL}(\mathcal{P}(V), \mathcal{W}_{\leq m+1})$$

is naturally identified with $\mathcal{P}(W) \times \mathrm{BL}(\mathcal{P}(V/W), \mathcal{W}_{\leq m}^W)$. In particular by (5.11) it is a smooth connected Cartier divisor. It follows that the exceptional locus of p is the disjoint union of these divisors. Thus p factors through the proscribed blowup, and the induced map to the blowup will have no exceptional divisors and is thus an isomorphism (as domain and image are smooth). \square

5.15. DEFINITION. For a subset $\sigma \subset Y$, consider the intersection

$$\tilde{\mathcal{P}}(\bar{\sigma}) := \bigcap_{M \in \sigma} \tilde{\mathcal{P}}(\overline{M}) \subset \mathbb{S}_Y.$$

5.16. PROPOSITION. $\tilde{\mathcal{P}}(\bar{\sigma})$ is non-empty iff σ is a simplex (4.3.1). Consider the convex subset $\text{Res}_\sigma(Y)$, a collection of convex subsets $\mathcal{W}_i \subset \text{Gr}(M_i/M_{i-1})$. There is a canonical identification

$$\tilde{\mathcal{P}}(\bar{\sigma}) = \prod_{m \geq i \geq 1} \text{BL}(\mathcal{P}(M_i/M_{i-1}), \mathcal{W}_i) =: \text{BL}(\mathcal{P}(\bar{\sigma}), \text{Res}_\sigma Y).$$

Proof. By (5.8) the k -points of the intersection are exactly those for which all $M \in \sigma$ are maximal. Thus if it is non-empty, σ is a simplex by (5.4). The expression for the intersection is immediate from the definition of Res (see (4.6)), and the functorial definitions of $\tilde{\mathcal{P}}(\bar{M})$ and BL . \square

5.17. REMARK. Observe by (5.10)–(5.16) that the special fibre S_Y has normal crossings. Moreover it can be canonically defined purely in terms of the subcomplex $Y \subset \mathcal{B}$. Indeed by (5.10) its irreducible components and their intersections are encoded by the $\text{BL}(\mathcal{P}(\bar{\sigma}), \text{Res}_\sigma(Y))$ for simplicies $\sigma \subset Y$, and by (4.10) we have canonical identifications

$$\text{Res}_\sigma Y = \text{Star}_\sigma Y \subset \text{Star}_\sigma \mathcal{B} = \text{Star}_{\bar{\sigma}} \text{Gr}(\bar{\Lambda}_m).$$

5.18. DEFINITION. Let $\sigma \subset Y$ be a simplex. Let $U(\sigma) \subset S_Y$ be the open subset whose complement is the closed subset of the special fibre given by the union of $\tilde{\mathcal{P}}(\bar{N})$, $[N] \in Y \setminus \sigma$.

5.19. LEMMA. $U(\sigma)$ is the union of the generic fibre together with the open subset of the special fibre consisting of all k -points whose simplex of maximal lattices (5.4) is contained in σ . It represents the following subfunctor: Let σ be the simplex (4.3.1). For $[M] \in Y$ choose minimal i so that $M^{M_m} \subset M_i$. A T -point of S_Y is a point of $U(\sigma)$ iff the composition

$$M_T \rightarrow (M_i)_T \twoheadrightarrow L((M_i)_T)$$

is surjective for all $[M] \in Y$.

Proof. Immediate from (5.8) and the definitions. \square

Note by (5.4) that the $U(\sigma)$ for $\sigma \subset Y$ give an open cover of S_Y . Faltings proves $U(\sigma)$ is non-singular, and semi-stable over $\text{Spec}(R)$, by writing down explicit local equations, [Fa, pg 167]. This can also be seen from the following:

5.20. PROPOSITION. Let $\sigma \subset Y$ be the simplex (4.3.1). Let $U \subset \mathcal{P}(M_m)$ be the open subset of quotients $M_m \rightarrow L$ such that $N^{M_m} \rightarrow L$ is surjective for all $[N] \in Y \setminus \sigma$.

Let $q : \text{BL}(\mathcal{P}(M_m), \bar{\sigma}) \rightarrow \mathcal{P}(M_m)$ be the iterated blowup of $\mathcal{P}(M_m)$ along the flag of subspaces of its special fibre

$$\mathcal{P}(M_m/M_{m-1}) \subset \mathcal{P}(M_m/M_{m-2}) \cdots \subset \mathcal{P}(M_m/M_1) \subset \mathcal{P}(M_m/M_0) = \mathcal{P}(\bar{M}_m)$$

i.e. blowup first the subspace

$$\mathcal{P}(M_m/M_{m-1}) \subset \mathcal{P}(\bar{M}_m) \subset \mathcal{P}(M_m)$$

then the strict transform of $\mathcal{P}(M_m/M_{m-2})$ etc. There is a natural isomorphism

$$U(\sigma) \rightarrow q^{-1}(U).$$

5.21. REMARK. When $\sigma = [M]$, (5.20) is immediate from (5.19). As this is the only case of (5.20) that we will need, we omit the proof, which in any case is analogous to (and simpler than) that of (5.11) and (5.14). (5.20) can also be deduced from the claim on [Fa, pg 168] that for any $[N] \in Y$ the natural map $\mathbb{S}_Y \rightarrow \mathcal{P}(N)$ is a composition of blowups with smooth centers (which Faltings describes).

Now fix \mathcal{F} as in (1.6.1).

5.22. LEMMA. *Let $Z \subset N$ be a subset with $|Z| = r + 1$. There is a unique stable lattice $[\Lambda_Z] \in [\mathcal{F}]$ such that the limits f_Z^Λ are generic (i.e. any r of them is an R -basis). In particular, there are finitely many stable lattices and Stab is finite. If we reorder so that $Z = \{0, 1, \dots, r\}$ and express*

$$f_0 = z^{a_1} p_1 f_1 + \dots z^{a_r} p_r f_r$$

with $a_i \in \mathbb{Z}$, $p_i \in R^$, then $\Lambda_Z = Rz^{a_1} f_1 + \dots Rz^{a_r} f_r$.*

Proof. It's clear that for Λ_Z as given, the limits \mathcal{F}^{Λ_Z} are in general position, so Λ_Z is stable.

For uniqueness, assume the limits \mathcal{F}_Z^Λ are in general position. Then f_i^Λ , $r \geq i \geq 1$ are an R -basis of Λ . Define $b_i \in \mathbb{Z}$ by $z^{b_i} f_i = f_i^\Lambda$. Scaling Λ we may assume $b_i \geq a_i$, with equality for some $r \geq i \geq 1$. Thus $f_0 = f_0^\Lambda$. Then $b_i = a_i$, for all i , for otherwise f_0^Λ will be in the span of some proper subset of the f_i^Λ , $r \geq i \geq 1$. So $\Lambda = \Lambda_Z$. \square

5.23. NOTATION. For a subset $I \subset N$ let $V_I \subset V_K$ be the vector subspace spanned by $f_i, i \in I$, and let $V^I := V/V_I$. For each lattice $M \subset V$, let M^I be its image in V^I , i.e. $M^I := M/M \cap V_I$.

Let $Y \subset [\mathcal{F}]$ be a finite convex collection, containing Stab . One checks immediately that the collection of equivalence classes

$$Y^I := \{[M^I]\}_{[M] \in Y}$$

is convex.

5.24. DEFINITION–LEMMA. *Let $\mathbb{B}_i \subset \mathbb{S}_Y$ be the subfunctor of compatible quotients such that q_M vanishes on f_i^M for all $[M] \in Y$.*

Then \mathbb{B}_i is the Deligne scheme for $Y^{\{i\}}$. $\mathbb{B}_i \subset \mathbb{S}_Y$ is non-singular and is the closure of the hyperplane on the generic fibre

$$\{f_i = 0\} \subset \mathcal{P}(V_K) \subset \mathbb{S}_Y.$$

Proof. Clearly $M \cap V_I = f_i^M R$, so it's clear $\mathbb{B}_i = \mathbb{S}_{Y^{\{i\}}}$. The rest now follows from (5.3). \square

5.25. PROPOSITION. *Let $[M] \in [\mathcal{F}]$ be a maximal lattice for a k point of*

$$\cap_{i \in I} \mathbb{B}_i \subset \mathbb{S}_Y.$$

Then the limits $f_i^M, i \in I$ are independent over R , i.e. they generate an R direct summand of M of rank $|I|$.

Proof. We consider the corresponding simplex of maximal lattices

$$zM = M_0 \subset M_1 \cdots \subset M_m = M.$$

For each $m \geq s \geq 1$, let $I_s \subset I$ be those i so that $f_i^M \in M_s \setminus M_{s-1}$. Clearly it is enough to show that the images of f_i^M , $i \in I_s$, in M_s/M_{s-1} are linearly independent. By scaling (which allows us to move any of the M_i to the M_1 position) it is enough to consider $s = 1$, and show that the images of f_i , $i \in I_1$ are linearly independent in M/zM . Suppose not. Choose a minimal set whose images are linearly dependent, which after reordering we may assume are f_0, f_1, \dots, f_p . Further reordering we may assume

$$f_1^M, f_2^M, \dots, f_r^M$$

are an R -basis of M , or equivalently, their images given a basis of M/zM . Renaming the f_i we can assume $f_i^M = f_i$. Now consider the unique expression

$$(5.25.1) \quad f_0 = \sum_{i=1}^r z^{a_i} p_i f_i$$

with $a_i \in \mathbb{Z}$, $p_i \in R^*$. By construction $a_i \geq 0$. Since the images of f_0, \dots, f_p in M/zM are a minimal linearly dependent set, it follows that $a_i \geq 1$ for $i > p$, and $a_i = 0$ for $p \geq i \geq 1$. Now let

$$\Lambda := Rf_1 + \dots + Rf_p + Rz^{a_{p+1}}f_{p+1} + \dots + Rz^{a_r}f_r.$$

Note $f_i^{M_1} = f_i$ for $p \geq i \geq 0$, so by assumption q_{M_1} vanishes on these f_i . $z^{a_t}f_t \in zM_k = M_0 \subset M_1$ for $t \geq p+1$. Thus by (5.2), q_{M_1} vanishes on these $z^{a_t}f_t$. Thus q_{M_1} vanishes on Λ . But by equation (5.25.1) and (5.22), Λ is stable. In particular $[\Lambda] \in Y$. Clearly $\Lambda \subset M_1$, but $\Lambda \not\subset M_0$. So the vanishing of $q_{M_1}|_{\Lambda}$ violates (5.4). \square

5.26. THEOREM. *Let $Y \subset [\mathcal{F}]$ be a finite convex set containing all the stable lattices.*

For any subset $I \subset N$, the scheme theoretic intersection

$$\bigcap_{i \in I} \mathbb{B}_i \subset \mathbb{S}_Y$$

is non-singular and (empty or) codimension $|I|$ and represents the Deligne functor \mathbb{S}_{Y^I} . The divisor $\mathbb{S}_Y + \mathbb{B} \subset \mathbb{S}_Y$ has normal crossings.

Proof. By (5.3) it's enough to show the intersection represents the Deligne functor.

Let \mathcal{I} be the intersection. It is obvious from the definitions and (5.24) that \mathcal{I} represents the subfunctor of \mathbb{S}_Y of compatible quotients q_M which vanish on f_i^M , $i \in M$, while \mathbb{S}_{Y^I} is the subfunctor where q_M vanishes on $M \cap V_I$. Since $f_i^M \in M \cap V_I$, clearly

$$\mathbb{S}_{Y^I} \subset \mathcal{I}$$

is a closed subscheme. Since by (5.3), \mathbb{S}_{Y^I} is flat over R , it's enough to show that they have the same special fibres. And so it is enough to show the subfunctors agree on $T = \text{Spec}(B)$ for B a local $k = R/zR$ algebra, with residue field k . By maximal for such a T -point, we mean maximal for the closed point. We take a family of compatible quotients vanishing on all f_i^M and show they actually vanish on $V_I \cap M$. By (5.4) it's enough to consider maximal M . But by (5.25) f_i^M , $i \in I$ are independent over R , so in fact they give an R -basis of $V_I \cap M$. \square

§6. GLOBAL GENERATION

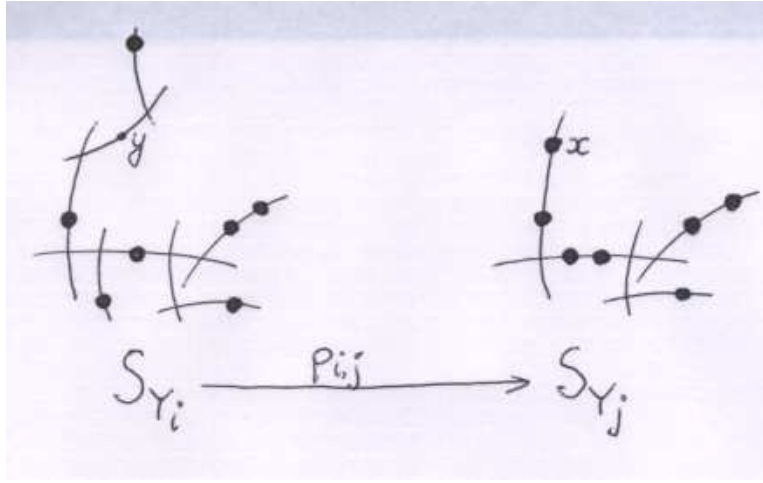
Here we complete the proof of (1.9).

6.1. BUBBLING. We choose an increasing sequence of finite convex subsets $Y_i \subset \mathcal{B}$ whose union is the membrane $[\mathcal{F}]$ – the existence for example follows from (4.11). We assume $\text{Stab} \subset Y_i$. We have the natural forgetful maps

$$p_{i,j} : \mathbb{S}_{Y_i} \rightarrow \mathbb{S}_{Y_j}, \quad i \geq j.$$

By (5.26), we have the vector bundle $\Omega_p^1(\log \mathbb{B})$ on \mathbb{S}_{Y_i} , see §9.

6.2. PROPOSITION. *Given a closed point $x \in \mathbb{S}_{Y_j}$ for all i sufficiently large there is a closed point $y \in \mathbb{S}_{Y_i} \setminus \mathbb{B}$ so that $p_{i,j}(y) = x$:*



Proof. Say x lies on $\tilde{\mathcal{P}}_{Y_j}(\overline{M})$, $[M] \in Y_j$. Choose i sufficiently big so that

$$[M + z^{-1}f^M R] \in Y_i$$

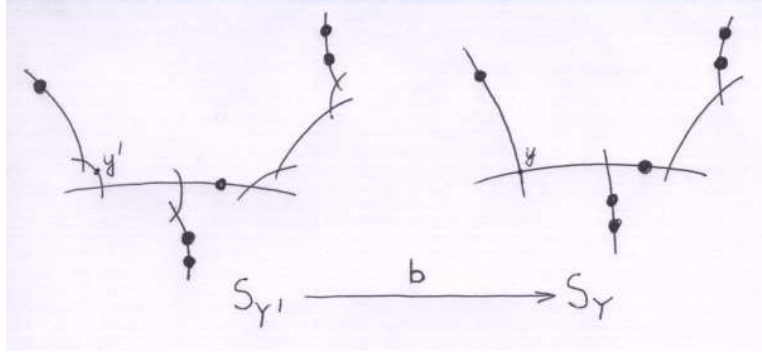
for all $f \in \mathcal{F}$. Clearly $p_{i,j}(\tilde{\mathcal{P}}_{Y_i}(\overline{M})) = \tilde{\mathcal{P}}_{Y_j}(\overline{M})$. It remains to show that $\tilde{\mathcal{P}}_{Y_j}(\overline{M})$ is disjoint from $\mathbb{B} \subset \mathbb{S}_{Y_j}$. Suppose that $q \in \tilde{\mathcal{P}}_{Y_j}(\overline{M}) \cap \mathbb{B}_k$. Let $N = M + z^{-1}f_k^M R$. Clearly $z^{-1}f_k^M = f_k^N$, so by definition of \mathbb{B}_k , q_N vanishes on $z^{-1}f_k^M$. But q_N vanishes on M by definition of a maximal lattice. So $q_N = 0$, a contradiction. \square

6.3. BARYCENTRIC SUBDIVISION TRICK. Next we introduce a convenient operation: Let $R' = k[[z^{1/m}]]$ and $\text{Spec}(R') \rightarrow \text{Spec}(R)$ the associated finite map. Let M_v be a collection of lattices in V_K , and Y their convex hull. Let $M'_v := M_v \otimes_R R'$, and let Y' be their convex hull.

6.4. PROPOSITION. *There is a commutative diagram*

$$\begin{array}{ccc} \mathbb{S}_{Y'} & \xrightarrow{b} & \mathbb{S}_Y \\ p' \downarrow & & \downarrow p \\ \text{Spec}(R') & \longrightarrow & \text{Spec}(R) \end{array}$$

with all arrows proper. If $m \geq r$ then given a k -point $y \in \mathbb{S}_Y$ there is a k -point $y' \in \mathbb{S}_{Y'}$ in its inverse image that lies on a unique irreducible component of the special fibre $S_{Y'}$:



Proof. For any R -object X , we denote by X' the base change to R' . It is clear that \mathbb{S}'_Y represents the functor $\mathbb{S}_{\tilde{Y}}$ defined as in (5.3) but for the non-convex collection $\tilde{Y} = \{M' \mid M \in Y\}$. Since $\tilde{Y} \subset Y'$, there is a forgetful map $\mathbb{S}_{Y'} \rightarrow \mathbb{S}'_Y$ sending compatible collections of quotients to compatible collections. This implies the commutative diagram above.

Now let $m \geq r$, $\omega = z^{1/m}$, and let $y \in \mathbb{S}_Y$ be a k -point with the simplex of maximal lattices

$$\sigma = \{M_0 = zM_k \subset M_1 \subset \dots \subset M_k\}.$$

By (5.6), y is determined by a collection of hyperplanes $H_i \subset M_i/M_{i-1}$ which do not contain any $\text{Res}_\sigma M$ for $M \in Y$. Let N be the R' lattice

$$N = M'_1 + \omega M'_2 + \omega^2 M'_3 + \dots + \omega^{k-1} M'_k.$$

Observe: $\omega^i M'_i \subset \omega N$, $k \geq i \geq 0$. ($M'_0 = \omega^{m-k} \omega^k M'_k$ and for $k > 0$ the inclusion is clear). Thus we have a map

$$(6.4.1) \quad \bigoplus_{i=1}^k M'_i/M'_{i-1} \xrightarrow{\oplus \omega^{i-1}} N/\omega N.$$

(6.4.1) is clearly surjective, thus it is an isomorphism, as domain and range are r -dimensional k -vector spaces. By the injectivity of the map we have

$$(6.4.2) \quad \omega^{i-1} M'_i \cap \omega N = \omega^{i-1} M'_{i-1}.$$

Now let $y' \in \mathcal{P}(N/zN)$ be given by any hyperplane $H' \subset N/zN$ which restricts to H_i on M_i/M_{i-1} under (6.4.1). Its enough to show $y' \in U([N])$ for then clearly y' is sent to y and N is the only maximal lattice of y' (or equivalently, by (5.8), y' lies on a unique irreducible component of the special fibre $S_{Y'}$). By (5.6) it's enough to show H' does not contain $\text{Res}_{[N]}[\Lambda]$ for $[\Lambda] \in Y'$, and by (4.10) its enough to check this for $[\Lambda] = [M']$, $[M] \in Y$.

We can assume $M = M^{M_k}$, $M \subset M_i$, $M \not\subset M_{i-1}$. Then $\text{Res}_\sigma[M] = (M + M_i)/M_{i-1}$ and it follows from (6.4.2) that $(M')^N = \omega^{i-1} M'$ thus we have

$$(\cdot \omega^{i-1})(\text{Res}_\sigma[M]) = \text{Res}_{[N]}[M'].$$

by (6.4.1). Thus H' does not contain $\text{Res}_{[N]}[M']$, since $H_i \not\supset \text{Res}_\sigma[M]$. \square

6.5. THEOREM. *Under the natural maps $p_{i,j} : \mathbb{S}_{Y_i} \rightarrow \mathbb{S}_{Y_j}$ for $j \leq i$, there is an induced isomorphism of vector bundles*

$$p_{i,j}^*(\Omega_p^1(\log \mathbb{B})) \rightarrow \Omega_p^1(\log \mathbb{B}).$$

Furthermore the global sections

$$\mathrm{dlog}(f/g), \quad f, g \in \mathcal{F}$$

generate $\Omega_p^1(\log \mathbb{B})$ globally. In particular $\omega_p(\mathbb{B})$ is globally generated.

Proof. Since the sections are pulled back from \mathbb{S}_{Y_j} , the last remark will imply the first. Furthermore, to prove the given sections generate at some point $y \in \mathbb{S}_{Y_j}$, it is enough to prove they generate at some inverse image point on \mathbb{S}_{Y_i} . Thus by (6.2) it's enough to prove they generate at a point $y \in \mathbb{S}_i \setminus \mathbb{B}$.

By (6.4), we have the proper (generically finite) map

$$\mathbb{S}_{Y'_i} \rightarrow \mathbb{S}_{Y_i}.$$

Clearly the $\mathrm{dlog}(f/g)$ pullback to the analogous forms on the domain, so by (6.4) we may assume y has a unique maximal lattice M . Now by (5.21) the natural map

$$\mathbb{S}_{Y_i} \rightarrow \mathcal{P}(M)$$

is an isomorphism of $y \in U([M])$ onto an open subset of $\mathcal{P}(M)$ which misses all of the hyperplanes

$$f_i^M = 0.$$

Note \mathcal{F}^M contains an R -basis of M . Reordering, say f_k^M , $r \geq k \geq 1$ give such a basis. Then it's enough to show the (regular) forms

$$\mathrm{dlog}(f_k^M / f_1^M), r \geq k \geq 2$$

give trivialisation of the ordinary cotangent bundle over the open set in question, which is obvious. \square

6.6. MINIMAL MODEL. By (6.5) the line bundle $\omega_p(\mathbb{B})$ is globally generated. We consider the p -relative minimal model, $\pi : \mathbb{S}_Y \rightarrow \bar{\mathbb{S}}$, i.e.

$$(6.6.1) \quad \bar{\mathbb{S}} := \mathrm{Proj} \bigoplus_m p_*(\omega_p(\mathbb{B})^{\otimes m}).$$

Note by (6.5) that $\bar{\mathbb{S}}$ is independent of Y . Let $\pi_*(\mathbb{B}_i) =: \mathbb{B}_i \subset \bar{\mathbb{S}}$.

6.7. THEOREM. *Let $\mathrm{Spec}(R) \rightarrow \bar{X}(r, n)$ be the unique extension of the map which sends the generic fibre to*

$$[\mathbb{P}^{r-1}, L_1 + \dots + L_n] \in \mathbb{P}^0(r, n) / \mathrm{PGL}_r = X(r, n) \subset \bar{X}(r, n).$$

The pullback of the universal visible contour family $(\mathcal{S}, \mathcal{B})$, (2.14) is $(\bar{\mathbb{S}}, \mathbb{B})$.

Proof. By (6.5) we have a natural surjection

$$V_n \otimes \mathcal{O}_{\mathbb{S}_Y} \rightarrow \Omega_p^1(\log \mathbb{B})$$

inducing a regular map $\mathbb{S}_Y \rightarrow G(r-1, n-1)$, which on the general fibre is Kapranov's visible contour embedding of \mathbb{P}^{r-1} given by the bundle of log forms with poles on the n general hyperplanes. This induces a regular map $\mathbb{S}_Y \rightarrow \mathcal{S}$ where $\mathcal{S} \rightarrow \mathrm{Spec}(R)$ is the pullback of the visible contour family, (2.14). $\omega_p(\mathbb{B})$ is pulled back from a relatively very ample line bundle (the

Plücker polarisation) on \mathcal{S} , by (2.19). Thus $\mathbb{S}_Y \rightarrow \mathcal{S}$ factors through a finite map

$$\overline{\mathbb{S}} \rightarrow \mathcal{S}.$$

The map is birational, an isomorphism on the generic fibre. By (2.15), \mathcal{S} is normal. Thus it is an isomorphism. \square

§7. BUBBLE SPACE

7.1. Here we prove (1.23). In §6 we choose finite convex $Y \subset [\mathcal{F}]$ containing Stab . Though there is a canonical choice, namely the convex hull of Stab , more esthetic is to take the infinite set $[\mathcal{F}]$. Let Y_i be an increasing sequence of finite convex subsets, containing Stab , with union $[\mathcal{F}]$. Call Y_i *full* along the simplex $\sigma \subset Y_i$ if $\text{Star}_{[M]} Y_i = \text{Star}_{[M]} [\mathcal{F}]$ for all $[M] \in \sigma$. It's clear that if Y_i is full along σ , so is Y_j for $j \geq i$. Let $U_i \subset Y_i$ be the union of all $U_{Y_i}(\sigma)$ such that Y_i is full along σ . The next lemma shows that we may view U_i as an increasing sequence of open sets. We define $\mathbb{S} = \cup_i U_i$.

7.2. LEMMA. *If Y_i is full along σ then $p_{ji}^{-1}(U_{Y_i}(\sigma)) \subset U_{Y_j}(\sigma)$. Moreover, $p_{ji}^{-1}(U_i) \subset U_j$, and the map $p_{ji}^{-1}(U_i) \rightarrow U_i$ is an isomorphism.*

Proof. Assume that Y_i is full along σ . Take $x \in p_{ji}^{-1}(U_{Y_j}(\sigma))$ and a maximal lattice $[N] \in Y_j$ for x . Maximal lattices form a simplex, so $[N]$ is adjacent to a lattice in σ and therefore $[N] \in Y_i$ because Y_i is full along σ . Now it's clear $[N]$ is maximal for $p_{ji}(x)$, so $[N] \in \sigma$. Thus $x \in U_{Y_i}(\sigma) \subset U_i$.

To show that $p_{ji}^{-1}(U_i) \rightarrow U_i$ is an isomorphism it suffices to check that $p_{ji}^{-1}(U_{Y_i}(\sigma)) \rightarrow U_{Y_i}(\sigma)$ is an isomorphism for any $\sigma \subset Y_i$. This map is proper and birational, and domain and range are non-singular, so to show it is an isomorphism, it's enough to check there are no-exceptional divisors, and so to check that each irreducible component of the special fibre of the domain maps onto an irreducible component of the special fibre for the image. These components are the (appropriate open subsets of the) $\mathcal{P}_{Y_j}(\overline{M})$, $[M] \in \sigma$, and it's obvious that $\mathcal{P}_{Y_j}(\overline{M}) \twoheadrightarrow \mathcal{P}_{Y_i}(\overline{M})$. \square

7.3. THEOREM. *\mathbb{S} is non-singular, and locally of finite type. Its special fibre, S_∞ , has smooth projective irreducible components and normal crossings. Let $\mathbb{B}_i \subset \mathbb{S}$ be the hyperplane $f_i = 0$ of the generic fibre. $\mathbb{B}_i \subset \mathbb{S}$ is closed and disjoint from S_∞ . $\mathbb{B} = \sum \mathbb{B}_i$ has normal crossings. In particular there is a natural vector bundle $\Omega_{\mathbb{S}/R}^1(\log \mathbb{B})$ whose determinant is $\omega_{\mathbb{S}/R}(\mathbb{B})$. The bundle is globally generated.*

There are natural surjective maps $p_i : \mathbb{S} \rightarrow \mathbb{S}_{Y_i}$ for all i , and natural isomorphisms

$$p_i^{-1}(\Omega_{\mathbb{P}}^1(\log \mathbb{B})) = \Omega_{\mathbb{S}/R}^1(\log \mathbb{B})$$

$$H^0(\mathbb{S}_{Y_i}, \Omega_{\mathbb{S}_i/R}^1(\log \mathbb{B})) \rightarrow H^0(\mathbb{S}, \Omega_{\mathbb{S}/R}^1(\log \mathbb{B})).$$

The differential forms $d\log(f/g)$, $f, g \in \mathcal{F}$ define a natural inclusion

$$V_n \subset H^0(S_\infty, \Omega_{S_\infty}^1).$$

The sections generate the bundle. The associate map $S_\infty \rightarrow G(r-1, n-1)$ factors through S_{Y_i} for all i , and its image is the special fibre of the pullback of the family $\mathcal{S} \rightarrow \overline{X}(r, n)$ for the map $\text{Spec}(R) \rightarrow \overline{X}/H$ given by \mathcal{F} .

Proof. Arguing as in (6.2), one can show that the special fibre of $U_i \rightarrow \operatorname{Spec}(R)$ is disjoint from $\mathbb{B} \subset \mathbb{S}_{Y_i}$. It follows that S_∞ is disjoint from \mathbb{B} .

We check that $\mathbb{S} \rightarrow \mathbb{S}_{Y_i}$ is surjective for all i . The rest then follows easily from (6.5) and (6.7). Take $[M] \in Y_i$. It's clear from the definitions that $\tilde{\mathcal{P}}(\overline{M}) \subset \mathbb{S}_{Y_j}$ surjects onto $\tilde{\mathcal{P}}(\overline{M}) \subset \mathbb{S}_{Y_i}$ for $j \geq i$. Moreover, there are only finitely many simplicies of $[\mathcal{F}]$ that contain $[M]$, by (4.7), and for j large Y_j will contain them all, from which it follows that $\tilde{\mathcal{P}}(\overline{M}) \subset U_j$. Thus the image of $\mathbb{S} \rightarrow Y_i$ will contain $\tilde{\mathcal{P}}(\overline{M})$. The union of the $\tilde{\mathcal{P}}(\overline{M})$ is the full special fibre so $\mathbb{S} \rightarrow Y_i$ is surjective. \square

7.4. The Deligne functor for $[\mathcal{F}]$ is not represented by a scheme. However, \mathbb{S} represents a natural subfunctor. In particular, \mathbb{S} is independent on the choice of a sequence Y_i :

7.5. THEOREM. \mathbb{S} represents the subfunctor of the Deligne functor for $[\mathcal{F}]$ a T -valued point of which is a collection of compatible quotients such that each k' -point of T admits a maximal lattice.

Proof. Take a k -point of U_i . By (7.2) any lattice $[M] \in Y_i$ maximal in Y_i for k will be maximal in $[\mathcal{F}]$. It follows that U_i is a subfunctor of the functor in the statement, and thus \mathbb{S} is a subfunctor. For the other direction its enough to consider T the spectrum of a local ring, with residue field k' . Consider a T -point of the subfunctor in the statement. Note that in the proof of (5.6) the only place finiteness of Y is used is to establish the existence of a maximal lattice which here we assume. So the k' point has a simplex of maximal lattices, σ , satisfying (5.6). For i large, Y_i will be full along σ , and now it is clear that the quotients define a T -point of U_i , and thus of \mathbb{S} . \square

§8. LIMIT VARIETY

8.1. The matroid decomposition corresponding to the fiber of the visible contour family can be readily obtained from the power series, as we now describe. From the matroid decomposition one can describe the fibre using [La, 5.3]. We assume the reader is familiar with the general theory of variation of GIT quotient, VGIT. See e.g. [DH]. We note in particular that $\Delta(r, n)$ parameterizes PGL_r -ample line bundles on $\mathbb{P}(r, n)$ with non-empty semi-stable locus.

8.2. DEFINITION. Call a polarisation $L \in \Delta(r, n)$ on $\mathbb{P}(r, n)$ *generic* if there are no strictly semi-stable points.

8.3. For $[M] \in \mathcal{B}$, let C_M be the configuration of limiting hyperplanes

$$\{f^{\overline{M}} = 0\} \subset \mathcal{P}(\overline{M}), \quad f \in \mathcal{F}$$

and let $P_M \subset \Delta(r, n)$ be the matroid polytope of C_M (see (3.2)).

8.4. LEMMA. If $L \in \Delta(r, n)$ is a generic polarisation then there is a unique $[M] \in [\mathcal{F}]$ so that the configuration C_M is L -stable. $[M]$ is GIT stable.

Proof. Let Q be the GIT quotient of $\mathbb{P}(r, n)$ given by L . Q is a fine moduli space for L -stable configurations and carries a universal family, a smooth étale locally trivial \mathbb{P}^{r-1} bundle. \mathcal{F} gives a K -point of Q , which extends uniquely to an R -point. The pullback of the universal family will be trivial

over R (as R is Henselian), and so $\mathcal{P}(M)$ for some lattice $M \subset V_K$. It's clear the limit configuration (given by the image in Q of the closed point of R) is equal to C_M . As C_M has no automorphisms it follows that this configuration contains r hyperplanes in general position, and so $[M] \in [\mathcal{F}]$. The proof shows that if C_N, C_M are both L -stable, then the rational map $\mathcal{P}(N) \rightarrow \mathcal{P}(M)$ is a regular isomorphism (either is the pullback of the universal family over Q), which implies $[N] = [M]$. C_M has trivial automorphism group, so the final remark holds by definition. \square

8.5. THEOREM. *Let $x \in \overline{X}(r, n)$ be the limit point for the one parameter family given by \mathcal{F} . Assume x belongs to a stratum given by the matroid decomposition \underline{P} . Then the maximal dimensional polytopes of \underline{P} are precisely the P_M for which C_M has no automorphisms, i.e. M is GIT-stable.*

Proof. By [Ka] and the theory of VGIT, the matroid decomposition is obtained as follows: GIT equivalence determines a polyhedral decomposition of $\Delta(r, n)$. Chambers (interiors of maximal dimensional polytopes in the decomposition) correspond to polarisations with no strictly semi-stable points. For each such chamber, there is a corresponding GIT quotient, which is a fine moduli space for configurations of hyperplanes stable for this polarisation. The one parameter family has a unique limit in each such quotient, and in particular associated to each chamber we have a limiting configuration. Associated to the configuration is its matroid polytope, and the polytopes obtained in this way are precisely the facets of \underline{P} . Now by (8.2), if L is a polarisation in a chamber, then the limiting configuration is C_{M_L} for a unique $[M_L] \in [\mathcal{F}]$. Conversely, if we take $[M] \in [\mathcal{F}]$ so that C_M has no automorphisms, the polytope $P_M \subset \Delta(r, n)$ is maximal dimensional, see [La, 1.11]. General $L \in P_M$ will be generic (in the sense of (8.2)) and it's clear that C_M is L -stable, so $M = M_L$. \square

8.6. STRATIFICATION. The membrane $[\mathcal{F}]$ is by (4.13) a union of apartments. We have an apartment for each $I \subset N$, $|I| = r$, and thus for each vertex of $\Delta(r, n)$. We stratify $[\mathcal{F}]$ by apartments – with one stratum for each collection of vertices – those points which lie in these, but no other, apartments. It follows easily from (8.5) that the stratum is non-empty iff the collection are the vertices of some $P \in \underline{P}$, in which case the stratum consists of those $[M] \in [\mathcal{F}]$ (or more generally, rational points of the realization, see (4.14)) with $P_M = P$. The dimension of the stratum is the codimension of the polytope P in $\Delta(r, n)$. We write $[\mathcal{F}]^k$ for the union of k -strata. Note $[\mathcal{F}]^0$ is precisely the union of GIT stable lattices.

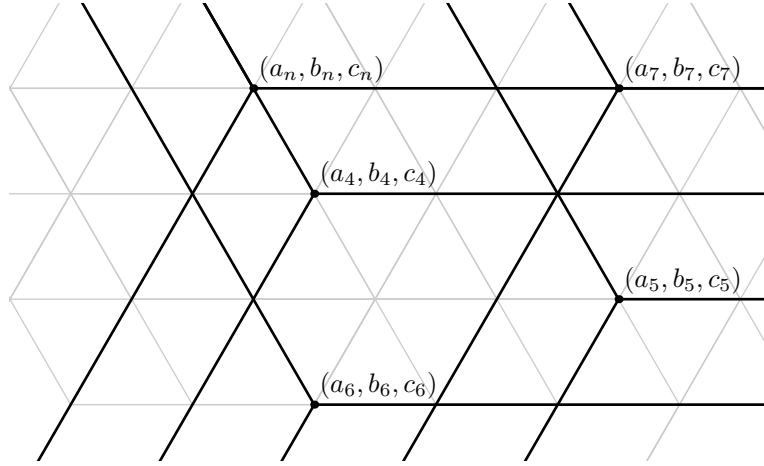
It is easy to describe the stratification in terms of the power series \mathcal{F} . Its enough to describe it in one apartment, say $[f_1, \dots, f_r]$:

For any a_1, \dots, a_r , let $S(a_1, \dots, a_r)$ be the stratification of $[f_1, \dots, f_r]$ by cones spanned by rays R_0, \dots, R_r where

$$R_i = [z^{a_1} f_1, \dots, z^{a_{i-1}} f_{i-1}, z^{a_i+p} f_i, z^{a_{i+1}} f_{i+1}, \dots, z^{a_r} f_r] \quad p \geq 0.$$

8.7. LEMMA. *Let $f_i = p_i^1 z^{a_i^1} f_1 + \dots + p_i^r z^{a_i^r} f_r$ for $i = r+1, \dots, n$, where $p_i^k \in R^*$. Then the stratification of $[f_1, \dots, f_r]$ is defined by intersections of the $S(a_i^1, \dots, a_i^r)$, see the picture for $r = 3$.*

Proof. Immediate from the definitions \square



Now consider the limit pair (S, B) . The irreducible components correspond to $[\mathcal{F}]^0$, for $[M] \in [\mathcal{F}]^0$ the corresponding component is the log canonical model of $\mathcal{P}(\overline{M}) \setminus C_{[M]}$ – the complement to the union of limiting hyperplanes. We note by [La, 5.3] that a collection of components have a common point of intersection iff the corresponding matroids have a common face, which is iff the points in $[\mathcal{F}]^0$ all lie on the boundary of the corresponding stratum. In particular, if they have non-empty intersection, they all lie in a single apartment.

8.8. LINES. From now on we assume that $r = 3$.

8.9. LEMMA. *A configuration of lines in \mathbb{P}^2 has trivial automorphism group iff it contains 4 lines in linear general position. A configuration has non-trivial automorphism group iff there is a point in the configuration which is in the complement of at most one of the lines. In this case the automorphism group is positive dimensional.*

Proof. This is easy linear algebra. \square

8.10. REMARK. By (8.9), stable is the same as GIT stable if $r = 3$. This fails in higher dimensions: the configuration of planes

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_1 + x_2 + x_3 = 0, x_2 + x_3 + x_4 = 0$$

in \mathbb{P}^3 is GIT stable but not stable.

8.11. LEMMA. *Let C be a stable configuration of lines indexed by N . Let $\tilde{S} \rightarrow \mathbb{P}^2$ be the blowup of all points of multiplicity at least 3. Let $B \subset \tilde{S}$ be the reduced inverse image of the lines. then $K_{\tilde{S}} + B$ is ample and*

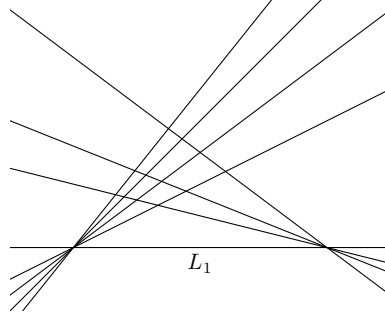
$$\mathbb{P}^2 \setminus C \subset \tilde{S}$$

is the log canonical compactification except in one case: If there are two points $a, b \in L$ on a line L of C such that any other line of C meets L in either a or b . In this case (the strict transform of) $L \subset \tilde{S}$ is a (-1) -curve, and the blowdown is $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\mathbb{P}^2 \setminus C \subset \mathbb{P}^1 \times \mathbb{P}^1$$

is the log canonical model, with boundary a union of fibers for the two rulings.

We refer to this exceptional case as a *special stable configuration*:



Proof. We induct on n . When $n = 4$ then $\tilde{S} = \mathbb{P}^2$ and the result is obvious. So we assume $n > 4$. If the configuration is special, the result is clear, so we assume it is not. By (8.9) we can drop a line, M , so the resulting configuration C' is stable. If C' is special, with special line L , then since C is not special, it follows that if we instead drop L , the resulting configuration is stable, and non-special. So we may assume C' is not special. Add primes to the notation to indicate analogous objects for C' . We have $q : \tilde{S} \rightarrow \tilde{S}'$, the blowup along the points of M where C has multiplicity exactly 3 (note $\tilde{S}' \rightarrow \mathbb{P}^2$ is an isomorphism around these points). Thus

$$K_{\tilde{S}} + B = q^*(K_{\tilde{S}'} + B') + M$$

(where we use the same symbol for a curve and for its strict transform). It's clear $K_{\tilde{S}} + B$ is q -ample. As $K_{\tilde{S}'} + B'$ is ample, the only curve on which $K_{\tilde{S}} + B$ can have non-positive intersection is M . But $(K_{\tilde{S}} + B) \cdot M > 0$ by adjunction, since C is not special. It follows that $K_{\tilde{S}} + B$ is ample. \square

8.12. THE LIMIT SURFACE. Now we describe the limit pair (S, B) precisely. The irreducible components are smooth, and described by (8.11). We write S_M for the component corresponding to $[M] \in \text{Stab}$. Unbounded 1-strata – rays in some apartment, correspond to irreducible components of B , bounded 1-strata correspond to irreducible components of $\text{Sing}(S)$. For each $[M] \in \text{Stab}$, the 1-strata which bound $[M]$ correspond to boundary components of S_M (components of the complement to $\mathcal{P}(\overline{M}) \setminus C_M$). S_M and S_N have one dimensional intersection iff $[M], [N]$ is the boundary of a 1-stratum, and in that case they are glued along the corresponding boundary component (a copy of \mathbb{P}^1). Triple points of S (points on three or more components) correspond to bounded 2-strata. The local analytic singularities of (S, B) are described by the following:

8.13. THEOREM. *Let $p \in S$ be a point where the pair (S, B) fails to have normal crossings. There are two possibilities for the germ of (S, B) in an analytic neighborhood $p \in U_p$:*

$$(8.13.1) \quad U_p = \langle e_1, e_2 \rangle \cup \langle e_2, e_3 \rangle \cup \langle e_3, e_4 \rangle \subset \mathbb{A}^4,$$

$B \cap U_p$ is the union of $\langle e_1 \rangle$ and $\langle e_4 \rangle$, and these are components of a single B_i .

$$(8.13.2) \quad U_p = \langle e_1, e_2 \rangle \cup \langle e_2, e_3 \rangle \cup \dots \cup \langle e_n, e_1 \rangle \subset \mathbb{A}^n,$$

$n = 3, 4, 5, 6$. $B \cap U_p = \emptyset$.

Here e_1, \dots, e_n are coordinate axes in \mathbb{A}^n , and $\langle \cdot \rangle$ is the linear span.

Proof. It is simple to classify bounded 2-strata using (8.7). Then the glueing among components is described by [La, 5.3]. \square

§9. THE BUNDLE OF RELATIVE LOG DIFFERENTIALS

Here we recall a general construction which we will use at various points throughout the paper:

Let

$$p : (\mathcal{S}, \mathcal{B}) \rightarrow C, \quad \mathcal{B} = \sum_{i=1}^n \mathcal{B}_i,$$

be a pair of a non-singular variety with normal crossing divisor, semi-stable over the curve C , in a neighborhood of $0 \in C$, i.e. $(\mathcal{S}, F + \mathcal{B})$ has normal crossings, where F is the fibre over 0. We assume the general fibre is projective, but not necessarily the special fibre. We define the bundle of relative log differentials $\Omega_p^1(\log \mathcal{B})$ by the exact sequence

$$(9.0.3) \quad 0 \longrightarrow \Omega_{C/k}^1(\log 0) \rightarrow \Omega_{\mathcal{S}/k}^1(\log F + \mathcal{B}) \rightarrow \Omega_p^1(\log \mathcal{B}) \rightarrow 0$$

Assume on the generic fibre S that the restrictions of the boundary components, B_i, B_j are linearly equivalent. Then we can choose a rational function f on \mathcal{S} so that

$$(f) = \mathcal{B}_i - \mathcal{B}_j + E$$

for E supported on F . Then $d\log(f)$ gives a global section of $\Omega_{\mathcal{S}/k}^1(\log F + \mathcal{B})$. Note f is unique up to multiplication by a unit on $C \setminus 0$, and thus the image of $d\log(f)$ in $\Omega_p^1(\log \mathcal{B})$, which we denote by $d\log(\mathcal{B}_i/\mathcal{B}_j)$ is independent of f .

From now on we assume that for the general fibre $H^0(S, \Omega_S^1) = 0$. $d\log(\mathcal{B}_i/\mathcal{B}_j)$ is now characterized as the unique section whose restriction to the general fibre has residue 1 along B_i , -1 along B_j and is regular off of $B_i + B_j$.

In this way we obtain a canonical map

$$(9.0.4) \quad V_n \rightarrow H^0(\mathcal{S}, \Omega_p^1(\log \mathcal{B}))$$

(V_n the standard k -representation of the symmetric group S_n) which is easily seen to be injective, e.g. by the description of the residues on the general fibre.

The restriction

$$(9.0.5) \quad \Omega_{\mathcal{S}/k}^1(\log B) := \Omega_p^1(\log \mathcal{B})|_S$$

for (S, B) the special fibre of $(\mathcal{S}, \mathcal{B})$, is canonically associated to (S, B) , i.e. is independent of the smoothing. See e.g. [Fr, §3] or [KN] – these authors treat normal crossing varieties without boundary, but the theory extends to normal crossing pairs in an obvious way. Finally there is a canonical residue map (e.g. induced via (9.0.3) by ordinary residues on \mathcal{S})

$$(9.0.6) \quad \text{Res} : \Omega_{\mathcal{S}/k}^1(\log B)|_{B_i} \rightarrow \mathcal{O}_{B_i}$$

together with (9.0.4) this gives a canonically split inclusion

$$(9.0.7) \quad V_n \subset H^0(S, \Omega_S^1(\log B)).$$

9.1. DEFINITION. Let (S, B) be a normal variety with boundary. Assume for an open subset $i : U \subset S$ with complement of codimension at least two that U is non-singular and $B|_U$ has normal crossings. Define

$$\Omega_{S/k}^1(\log B) := i_*(\Omega_{U/k}^1(\log B|_U)).$$

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